

# RECONSTRUCTIONS FROM BOUNDARY MEASUREMENTS ON ADMISSIBLE MANIFOLDS

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ABSTRACT. We prove that a potential  $q$  can be reconstructed from the Dirichlet-to-Neumann map for the Schrödinger operator  $-\Delta_g + q$  in a fixed admissible 3-dimensional Riemannian manifold  $(M, g)$ . We also show that an admissible metric  $g$  in a fixed conformal class can be constructed from the Dirichlet-to-Neumann map for  $\Delta_g$ . This is a constructive version of earlier uniqueness results by Dos Santos Ferreira et al. [7] on admissible manifolds, and extends the reconstruction procedure of Nachman [21] in Euclidean space. The main points are the derivation of a boundary integral equation characterizing the boundary values of complex geometrical optics solutions, and the development of associated layer potentials adapted to a cylindrical geometry.

## 1. INTRODUCTION

This paper is concerned with the problem of reconstructing material parameters of a medium from boundary measurements. A typical question of this type is Calderón's inverse conductivity problem [6], which consists in recovering the conductivity of a body from voltage to current measurements at the boundary. For bounded domains in Euclidean space in dimensions  $n \geq 3$ , it was proved in [30] that a smooth positive scalar conductivity  $\sigma$  is uniquely determined by the Dirichlet-to-Neumann map (DN map)  $\Lambda_\sigma$  representing the boundary measurements. This uniqueness result was then extended to a reconstruction procedure in [21] and independently in [24], see also [13]. In two dimensions, uniqueness and reconstruction for this problem was proved even for bounded measurable conductivities in [2], [3].

In this paper we consider Calderón's inverse problem and related questions in anisotropic media, where the conductivity depends on direction. This corresponds to replacing the scalar conductivity  $\sigma$  by a smooth symmetric positive definite matrix. The question then is to recover the matrix  $\sigma$  from the DN map  $\Lambda_\sigma$ , up to the natural obstruction given by diffeomorphisms which fix the boundary. If  $n = 2$ , it is proved in [1] that any bounded measurable matrix conductivity  $\sigma$  is determined by  $\Lambda_\sigma$  up to diffeomorphism. For constructive results see [15] and the references therein.

In three and higher dimensions the anisotropic Calderón problem is open even for smooth matrix conductivities. We refer to [7] for a more thorough discussion and references to known results. It was observed in [19] that the anisotropic Calderón problem is closely related to certain inverse problems for the Laplace-Beltrami operator on a Riemannian manifold, which we set out to define.

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**Statement of main results.** Let  $(M, g)$  be a compact oriented Riemannian manifold with  $C^\infty$  boundary, and let  $\Delta_g$  be the Laplace-Beltrami operator. In local coordinates

$$\Delta_g u = |g|^{-1/2} \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left( |g|^{1/2} g^{jk} \frac{\partial u}{\partial x_k} \right)$$

where  $g = (g_{jk})$  is the metric in local coordinates,  $(g^{jk})$  is the inverse matrix of  $(g_{jk})$ , and  $|g| = \det(g_{jk})$ . Consider the Dirichlet problem

$$\begin{cases} \Delta_g u = 0 & \text{in } M, \\ u = f & \text{on } \partial M. \end{cases}$$

For any  $f \in H^{3/2}(\partial M)$  there is a unique solution  $u \in H^2(M)$ , and the DN map is defined by

$$\Lambda_g : H^{3/2}(\partial M) \rightarrow H^{1/2}(\partial M), \quad f \mapsto \partial_\nu u|_{\partial M}$$

where the normal derivative is given by

$$\partial_\nu u|_{\partial M} = \sum_{j,k=1}^n g^{jk} \frac{\partial u}{\partial x_j} \nu_k.$$

Here  $\nu_k = \sum_{l=1}^n g_{kl} \nu^l$ , and  $(\nu^1, \dots, \nu^n)$  is the coordinate expression for the unit outer normal vector  $\nu$  on  $\partial M$ .

Our first result states that the map  $\Lambda_g$  constructively determines  $g$  within a known conformal class of admissible metrics (as defined below).

**Theorem 1.1.** Let  $(M, g)$  be a given admissible 3-dimensional manifold. If  $c$  is a smooth positive function on  $M$ , then from the knowledge of  $\Lambda_{cg}$  one can constructively determine  $c$ .

The next question concerns an inverse problem for the Schrödinger equation in  $(M, g)$ . If  $q$  is a smooth function on  $M$ , we consider the Dirichlet problem

$$\begin{cases} (-\Delta_g + q)u = 0 & \text{in } M, \\ u = f & \text{on } \partial M. \end{cases}$$

We make the standing assumption that

$$0 \text{ is not a Dirichlet eigenvalue of } -\Delta_g + q \text{ in } M.$$

This means that for any  $f \in H^{3/2}(\partial M)$  the equation has a unique solution  $u \in H^2(M)$ , and the DN map can be defined by

$$\Lambda_{g,q} : H^{3/2}(\partial M) \rightarrow H^{1/2}(\partial M), \quad f \mapsto \partial_\nu u|_{\partial M}.$$

The second main result is as follows.

**Theorem 1.2.** Let  $(M, g)$  be a given admissible 3-dimensional manifold. If  $q$  is a smooth function on  $M$ , then from the knowledge of  $\Lambda_{g,q}$  one can constructively determine  $q$ .

To complete the statement of the main results, let us give the definition of admissible manifolds. These arose in [7] as the first class of manifolds beyond real-analytic or Einstein ones for which one can prove uniqueness results for the anisotropic inverse problems described above.

**Definition.** A compact oriented Riemannian manifold  $(M, g)$  with smooth boundary is *admissible* if  $\dim(M) \geq 3$  and if  $(M, g) \subset\subset (T^{\text{int}}, g)$  where  $T = \mathbb{R} \times M_0$  is a cylinder with metric  $g = c(e \oplus g_0)$  (here  $c$  is a smooth positive function and  $e$  is the Euclidean metric on  $\mathbb{R}$ ), and  $(M_0, g_0)$  is an  $(n - 1)$ -dimensional simple manifold.

Thus, up to a conformal factor, an admissible manifold is embedded in a cylinder  $(\mathbb{R} \times M_0, e \oplus g_0)$  and therefore has a Euclidean direction. The transversal manifold  $(M_0, g_0)$  needs further to be simple:

**Definition.** A compact manifold  $(M_0, g_0)$  with boundary is *simple* if for any  $p \in M_0$  the exponential map  $\exp_p$  is a diffeomorphism from its maximal domain in  $T_p M_0$  onto  $M_0$ , and if  $\partial M_0$  is strictly convex (meaning that the second fundamental form is positive definite).

Examples of admissible manifolds include subdomains of the model spaces (Euclidean space, sphere minus a point, hyperbolic space), sufficiently small subdomains of any conformally flat manifold, and domains in  $\mathbb{R}^n$  equipped with a metric of the form

$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}$$

where  $c$  is a positive smooth function and where  $g_0$  is a simple metric in the  $x'$  variables.

The unique determination results corresponding to Theorems 1.1 and 1.2 were proved in [7], where also earlier work is discussed and further references are given. There are several recent results that are concerned with the two-dimensional case. If  $M$  is a domain in  $\mathbb{R}^2$ , [5] proved the uniqueness result corresponding to Theorem 1.2 and briefly discussed a reconstruction procedure. A proof with constructive character was given in [11] for arbitrary Riemann surfaces  $(M, g)$  with boundary (also for the magnetic Schrödinger operator), based on earlier nonconstructive proofs in [10], [17]. Various constructive results for the two-dimensional case, also on Riemann surfaces, appear in [12], [14], [15], [16], and for the three-dimensional case an improved reconstruction result is given in [25].

**Outline of proof.** As mentioned, the uniqueness results corresponding to Theorems 1.1 and 1.2 were proved in [7], but the proofs were not constructive. The main point of the present paper is to give constructive proofs, following the well-known reconstruction procedure of Nachman [21] in the case where  $M$  is a bounded domain in  $\mathbb{R}^n$  and  $g$  is the Euclidean metric. We do not make any claims about the practicality of the reconstruction procedure, but we do prove that all the steps in the corresponding uniqueness proofs in [7] can be carried out in a constructive way.

The argument in [21] involves *complex geometrical optics* (CGO) solutions  $u = e^{-\zeta \cdot x}(1+r)$  to the Schrödinger equation  $(-\Delta + q)u = 0$  in  $\mathbb{R}^n$ , and relies in a crucial way on a uniqueness notion for these solutions upon fixing decay at infinity. One has several equivalent ways of characterizing these solutions, and in particular it is possible to recover the boundary value  $u|_{\partial M}$  as the unique solution to a boundary integral equation on  $\partial M$  involving  $\Lambda_{g,q}$  and other known quantities.

CGO solutions on admissible Riemannian manifolds were constructed in [7] by Carleman estimates. The solutions were given in a compact manifold, and the construction did not involve a notion of uniqueness. The paper [18] introduced a direct Fourier analytic construction of CGO solutions, valid in the cylinder  $T$  and with a uniqueness notion obtained by fixing a decay condition in the Euclidean variable and Dirichlet boundary values on  $\partial T$ . We shall use the solutions constructed in [18] to prove Theorem 1.2 (which implies Theorem 1.1 after a simple reduction).

We next sketch the proof of Theorem 1.2. Let  $(M, g) \subset\subset (T^{\text{int}}, g)$  be an admissible manifold, and assume that  $g = e \oplus g_0$  where  $(M_0, g_0)$  is simple. Here we suppose, for simplicity, that  $c = 1$ . We assume that  $\Lambda_{g,q}$  and  $(M, g)$  are known (thus also  $\Lambda_{g,0}$  is known), and use the basic integral identity

$$\int_{\partial M} (\Lambda_{g,q} - \Lambda_{g,0})(u|_{\partial M})v \, dS = \int_M quv \, dV \quad (1.1)$$

which is valid for any solutions  $u, v \in H^2(M)$  of  $(-\Delta_g + q)u = 0$  and  $-\Delta_g v = 0$  in  $M$ . We take  $u$  and  $v$  to be suitable CGO solutions such that  $u|_{\partial M}$  may be obtained from  $\Lambda_{g,q}$  as the unique solution of a boundary integral equation, and  $v|_{\partial M}$  is explicitly given. Then the left hand side of (1.1) is known. Taking the limit as  $\tau \rightarrow \infty$  and varying certain parameters in the solutions, we recover in this way the quantity

$$\int_0^\infty e^{-2\lambda r} \left[ \int_{-\infty}^\infty e^{2i\lambda x_1} q(x_1, r, \theta) \, dx_1 \right] dr \quad (1.2)$$

for any  $\theta \in S^{n-2}$ , where  $\lambda$  is any nonzero real number and  $(r, \theta)$  are polar normal coordinates in  $M_0$  with center on  $\partial M_0$ . We have extended  $q$  into  $T \setminus M$  as a function in  $C_c^\infty(T^{\text{int}})$ .

Now, since  $(r, \theta)$  are polar normal coordinates, the curves  $\gamma : r \mapsto (r, \theta)$  are unit speed geodesics in  $(M_0, g_0)$  for any fixed  $\theta$ . Denoting the quantity in brackets in (1.2) by  $f_\lambda(r, \theta)$ , it follows that we have recovered

$$\int_\gamma e^{-2\lambda r} f_\lambda(\gamma(r)) \, dr$$

for any maximal geodesic  $\gamma$  going from  $\partial M_0$  into  $M_0$ , and for any  $\lambda \neq 0$ . This is the attenuated geodesic ray transform of  $f_\lambda$  with constant attenuation  $-2\lambda$ , see [7], [29]. For any  $\lambda$  such that this transform can be inverted, we recover  $f_\lambda$  which is just the one-dimensional Fourier transform

$$\mathcal{F}_{x_1}\{q(\cdot, x')\}(-2\lambda) \quad \text{for all } x' \in M_0.$$

It was proved in [7] and [8] that the attenuated ray transform is invertible for small  $|\lambda|$ , thus giving information on  $\mathcal{F}_{x_1}\{q(\cdot, x')\}$  for small frequencies. This determines the compactly supported function  $q(\cdot, x')$  uniquely by the Paley-Wiener theorem. To make this step more constructive one would like to invert the attenuated ray transform for all  $\lambda$ , which would yield  $q(\cdot, x')$  by taking the inverse Fourier transform. Up to now the argument has been valid for  $\dim(M) \geq 3$ . However, if  $\dim(M) = 3$  then  $(M_0, g_0)$  is a 2D simple manifold, and the recent result [28] shows that the attenuated ray transform is invertible for any attenuation. We can use the inversion procedure in [28] to conclude the reconstruction of  $q$  from  $\Lambda_{g,q}$  if  $(M, g)$  is 3-dimensional.

**Boundary integral equation.** The main new point in the proof is a Fredholm boundary integral equation characterizing the values on  $\partial M$  of suitable CGO solutions in  $T$ . This equation has the form

$$(\text{Id} + \gamma S_\tau(\Lambda_{g,q} - \Lambda_{g,0}))f = u_0 \quad \text{on } \partial M. \quad (1.3)$$

Here  $u_0$  is an explicit function on  $\partial M$  depending on various parameters,  $\gamma$  is the trace operator  $H^2(M) \rightarrow H^{3/2}(\partial M)$ , and  $S_\tau$  is a special single layer potential depending on a large parameter  $\tau$  and adapted to the CGO solutions and the geometry of the cylinder  $(T, g) = (\mathbb{R} \times M_0, e \oplus g_0)$ . In fact, we have

$$S_\tau f(x) = \int_{\partial M} K_\tau(x, y) f(y) dS(y), \quad x \in T^{\text{int}} \setminus \partial M,$$

where the integral kernel  $K_\tau(x, y)$  is explicitly determined by  $\tau$  and by the Dirichlet eigenvalues and eigenfunctions of the Laplace-Beltrami operator on  $(M_0, g_0)$ . We establish basic properties of the single layer operator in Section 2.

We prove that for suitable choices of  $u_0$  and for  $|\tau|$  large, the equation (1.3) has a unique solution  $f \in H^{3/2}(\partial M)$ , and one has  $f = u|_{\partial M}$  where  $u$  is the corresponding CGO solution. Since the operator on the left hand side of (1.3) is determined by the boundary measurements and since  $u_0$  is explicit, we can indeed determine the boundary values of CGO solutions by solving this Fredholm integral equation.

This approach is analogous to [21] which considers the Euclidean case, except that the uniqueness notion for CGO solutions is obtained from decay conditions and Dirichlet boundary values on the cylinder  $(T, g)$  instead of a decay condition at infinity in  $\mathbb{R}^n$ . Recently in [23] another constructive approach appeared. There the boundary integral equation is obtained via Carleman estimates in  $M$ , and no extension of  $M$  to a larger set is needed. It is presumable that a similar approach would work in our case. However, the single layer potential obtained from Carleman estimates is perhaps not so explicit as the operator  $S_\tau$  introduced above.

We remark here that it would be interesting to establish reconstruction results corresponding to Theorems 1.1 and 1.2 for the magnetic Schrödinger equation or for the time-harmonic Maxwell equations. Uniqueness results for these equations are proved in [7], [18], and constructive results in the Euclidean case appear in [26], [27].

**Structure of paper.** Section 1 is the introduction. The basic properties of the single layer operator  $S_\tau$  and related Faddeev Green functions are considered in Section 2. In Section 3 we introduce several equivalent ways of characterizing CGO solutions, including the required boundary integral equation. The results in Sections 2 and 3 are in fact valid for any transversal manifold  $(M_0, g_0)$  with smooth boundary (not necessarily simple). The proofs of Theorems 1.1 and 1.2 are given in Section 4.

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## 2. BOUNDARY LAYER POTENTIALS

**Notation and function spaces.** In this section we assume that  $(M, g)$  is a compact manifold with smooth boundary, having dimension  $n \geq 3$ , and that  $(M, g) \subset\subset (T^{\text{int}}, g)$  where  $T = \mathbb{R} \times M_0$  and  $g = e \oplus g_0$ , and  $(M_0, g_0)$  is any compact  $(n - 1)$ -dimensional manifold with boundary (no restrictions on the metric  $g_0$ ). Points of  $T$  are written as  $x = (x_1, x')$  where  $x_1$  is the Euclidean variable and  $x'$  is a point in the transversal manifold  $M_0$ .

We write  $\langle \cdot, \cdot \rangle$  for the inner product of tangent vectors, 1-forms, and other tensors, and  $|\cdot|$  for the norm. The volume element in  $(T, g)$  is

$$dV(x) = dV_g(x) = dx_1 dV_{g_0}(x')$$

with  $dV_{g_0}$  the volume element in  $(M_0, g_0)$ . We also write  $\Gamma = \partial M \subset\subset T$ , and denote by  $dS$  the volume element on  $\Gamma$ .

Let  $L^2(T) = L^2(T, dV)$  be the standard  $L^2$  space in  $T$ , and let  $H^s(T)$  be the corresponding  $L^2$  Sobolev spaces. Since  $M_0$  is compact, we define

$$H_{\text{loc}}^s(T) = \{f; f \in H^s([-R, R] \times M_0) \text{ for any } R > 0\},$$

$$H_c^s(T) = \{f \in H^s(T); f(x_1, x') = 0 \text{ when } |x_1| \geq R \text{ for some } R > 0\}.$$

Writing  $\langle t \rangle = (1 + t^2)^{1/2}$ , we introduce for  $\delta \neq 0$ ,  $s \geq 0$  the weighted spaces

$$L_\delta^2(T) = \{f \in L_{\text{loc}}^2(T); \langle x_1 \rangle^\delta f \in L^2(T)\},$$

$$H_\delta^s(T) = \{f \in H_{\text{loc}}^s(T); \langle x_1 \rangle^\delta f \in H^s(T)\},$$

$$H_{\delta,0}^1(T) = \{f \in H_\delta^1(T); f|_{\partial T} = 0\},$$

$$H_{\text{loc},0}^1(T) = \{f \in H_{\text{loc}}^1(T); f|_{\partial T} = 0\},$$

and

$$H_{-\infty,0}^1(T) = \bigcup_{\delta \in \mathbb{R}} H_{\delta,0}^1(T).$$

Also,  $H_0^1(T) = \{f \in H^1(T); f|_{\partial T} = 0\}$ . We define, in the  $L^2(T)$  duality,

$$H^{-1}(T) = (H_0^1(T))^*.$$

On  $\Gamma = \partial M$  we consider the usual space  $L^2(\Gamma) = L^2(\Gamma, dS)$  and the corresponding Sobolev spaces  $H^s(\Gamma)$ .

Let  $0 < \lambda_1 \leq \lambda_2 \leq \dots$  be the Dirichlet eigenvalues of  $-\Delta_{g_0}$  in  $(M_0, g_0)$ , and let  $\{\phi_l\}_{l=1}^\infty$  be an orthonormal basis of  $L^2(M_0)$  consisting of Dirichlet eigenfunctions,

$$-\Delta_{g_0} \phi_l = \lambda_l \phi_l \text{ in } M_0, \quad \phi_l \in H_0^1(M_0).$$

We write  $\text{Spec}(-\Delta_{g_0}) = \{\lambda_l\}_{l=1}^\infty$ . If  $f \in L^2(T)$  we consider the partial Fourier coefficients

$$\tilde{f}(x_1, l) = (f(x_1, \cdot), \phi_l)_{L^2(M_0)}.$$

The Parseval identity implies that

$$\|f\|_{L^2(T)}^2 = \int_{-\infty}^{\infty} \sum_{l=1}^{\infty} |\tilde{f}(x_1, l)|^2 dx_1.$$

**Standard single layer operator.** Let  $K_0$  be the usual inverse of the Dirichlet Laplacian on  $T$ , defined as follows: since any  $u \in H_0^1(T)$  satisfies the Poincaré inequality

$$\begin{aligned} \int_T |u|^2 dV &= \int_{-\infty}^{\infty} \int_{M_0} |u(x_1, x')|^2 dV_{g_0} dx_1 \\ &\leq \int_{-\infty}^{\infty} C \int_{M_0} |d_{x'} u(x_1, x')|^2 dV_{g_0} dx_1 \leq C \int_T |du|^2 dV, \end{aligned}$$

the bilinear form  $B(u, v) = \int_T \langle du, dv \rangle dV$  is bounded and coercive on  $H_0^1(T)$ . Consequently, for any  $F \in H^{-1}(T)$  there is a unique weak solution  $u = K_0 F$  in  $H_0^1(T)$  of the equation  $-\Delta_g u = F$  in  $T$ . Thus  $K_0$  is a bounded linear operator

$$K_0 : H^{-1}(T) \rightarrow H_0^1(T), \quad -\Delta_g K_0 = \text{Id}.$$

We also consider the trace operator which restricts functions to  $\Gamma$ ,

$$\gamma : H_0^1(T) \rightarrow H^{1/2}(\Gamma), \quad \gamma u = u|_{\Gamma}.$$

The adjoint of  $\gamma$  satisfies

$$\begin{aligned} \gamma^* : H^{-1/2}(\Gamma) &\rightarrow H^{-1}(T), \\ (\gamma^* h, \varphi)_{L^2(T)} &= (h, \gamma \varphi)_{L^2(\Gamma)} \quad \text{for any } \varphi \in H_0^1(T). \end{aligned}$$

Thus formally  $\gamma^* h = h dS$ .

**Definition.** The standard single layer operator on  $T$  is the map

$$S_0 = K_0 \gamma^* : H^{-1/2}(\Gamma) \rightarrow H_0^1(T).$$

The next result gives the basic jump and mapping properties of  $S_0$ . Here we write  $M_- = M^{\text{int}}$ ,  $M_+ = T \setminus M$ , and  $\gamma_{\mp} u = (u|_{M_{\mp}})|_{\Gamma}$  for the restriction of  $u$  to  $\Gamma$  from the interior or exterior. If  $u$  is a function on  $T \setminus \Gamma$  such that  $u|_{M_{\mp}}$  are  $H^1$  in  $M_{\mp}$  and satisfy  $-\Delta_g u = 0$  in  $M_{\mp}$ , we define the normal derivatives from the interior or exterior weakly as elements of  $H^{-1/2}(\Gamma)$  by

$$((\partial_{\nu} u)_{\mp}, h)_{L^2(\Gamma)} = \pm (d(u|_{M_{\mp}}), de_h)_{L^2(M_{\mp})}, \quad h \in H^{1/2}(\Gamma),$$

where  $e_h \in H^1(T)$  is any function with  $e_h|_{\Gamma} = h$  and  $e_h|_{\partial T} = 0$ . The jumps on  $\Gamma$  are defined by

$$\begin{aligned} [u]_{\Gamma} &= \gamma_- u - \gamma_+ u, \\ [\partial_{\nu} u]_{\Gamma} &= (\partial_{\nu} u)_- - (\partial_{\nu} u)_+. \end{aligned}$$

**Lemma 2.1.** If  $f \in H^{-1/2}(\Gamma)$  then  $u = S_0 f \in H_0^1(T)$  satisfies

$$\begin{aligned} -\Delta_g u &= 0 \quad \text{in } M_{\pm}, \\ [u]_{\Gamma} &= 0, \\ [\partial_{\nu} u]_{\Gamma} &= f. \end{aligned}$$

If  $f \in H^s(\Gamma)$  for  $s \geq -1/2$  then  $u|_{M_-} \in H^{s+3/2}(M_-)$  and  $u|_{M_+} = \tilde{u}|_{M_+}$  for some  $\tilde{u} \in H_{\text{loc}}^{s+3/2}(T) \cap H_0^1(T)$ .

*Proof.* Harmonicity and the first jump property are direct consequences of the properties of  $K_0$ . The definitions also imply that for  $h \in H^{1/2}(\Gamma)$ ,

$$([\partial_\nu u]|_\Gamma, h)_{L^2(\Gamma)} = (du, de_h)_{L^2(T)} = (\gamma^* f, e_h)_{L^2(T)} = (f, h)_{L^2(\Gamma)}.$$

This shows the second jump property. If  $f \in H^{k+1/2}(\Gamma)$  with  $k \geq 0$ , then  $u$  satisfies

$$\begin{aligned} -\Delta_g u &= 0 \quad \text{in } M_\pm, \\ [u]_\Gamma &= 0, \\ [\partial_\nu u]_\Gamma &= f \in H^{k+1/2}(\Gamma). \end{aligned}$$

The transmission property [20, Theorem 4.20] implies that  $u|_{M_\pm}$  are  $H^{k+2}$  near  $\Gamma$ . Then the properties for  $u$  follow from standard interior and boundary regularity for  $-\Delta_g$  and from interpolation.  $\square$

**Corollary 2.2.** The trace single layer potential satisfies for  $s \geq -1/2$

$$\gamma S_0 : H^s(\Gamma) \rightarrow H^{s+1}(\Gamma).$$

Since  $K_0$  maps  $C_c^\infty(T^{\text{int}})$  to  $L^2(T^{\text{int}})$ , the Schwartz kernel theorem shows that there is a distributional integral kernel  $K_0(x, y)$ . (We restrict to  $T^{\text{int}}$  to avoid having to talk about distributions on manifolds with boundary.) Then formally

$$S_0 f(x) = \int_\Gamma K_0(x, y) f(y) dS(y).$$

We will need some basic properties of the integral kernel. These are easily obtained by comparing to the Green function on compact manifolds [4], [31].

**Lemma 2.3.** The kernel  $K_0$  is smooth in  $T^{\text{int}} \times T^{\text{int}}$  away from the diagonal, and it satisfies (with  $d$  the Riemannian distance)

$$|K_0(x, y)| \leq C_U d(x, y)^{2-n}, \quad x, y \in \bar{U} \subset\subset T^{\text{int}}.$$

Also,  $K_0(x, y) = K_0(y, x)$ , and for any  $x \in T^{\text{int}}$  one has  $\Delta_g(K_0(x, \cdot)) = 0$  in  $T^{\text{int}} \setminus \{x\}$ .

*Proof.* The condition  $K_0(x, y) = K_0(y, x)$  follows since  $-\Delta_g$  is symmetric. Let  $U \subset\subset W \subset\subset T^{\text{int}}$  where  $U$  is open and  $(\bar{W}, g)$  is a compact manifold with smooth boundary, and let  $G(x, y)$  be the Dirichlet Green function for the Laplacian in  $(\bar{W}, g)$  [4, Section 4.2]. We will prove that

$$K_0(x, y) = G(x, y) + R(x, y), \quad x, y \in \bar{U}$$

where  $R \in C^\infty(\bar{U} \times \bar{U})$ . Since the function  $G(x, y)$  has the stated properties by [4, Theorem 4.17], they also follow for  $K_0$  by using smoothness of  $R$  and simple arguments.

Consider  $\varphi \in C_c^\infty(U)$  and let  $v = u_0 - u_1$  where  $u_0 = K_0 \varphi$  and

$$u_1(x) = \int_W G(x, y) \varphi(y) dV(y).$$

Then  $-\Delta_g u_1 = \varphi$  in  $W$  with  $u_1|_{\partial W} = 0$ . It follows that  $v \in H^1(W)$ ,  $\|v\|_{H^1(W)} \leq C \|\varphi\|_{L^2(U)}$ , and  $\Delta_g v = 0$  in  $W$ . By elliptic regularity and Sobolev embedding,

$$\|\nabla^k v\|_{L^\infty(U)} \leq C_k \|v\|_{L^2(W)} \leq C_k \|\varphi\|_{L^2(U)}.$$

This may be rewritten as

$$\left| \int_W \nabla_x^k R(x, y) \varphi(y) dV(y) \right| \leq C_k \|\varphi\|_{L^2(U)}.$$

Consequently  $\|\nabla_x^k R(x, \cdot)\|_{L^2(U)} \leq C_k$  uniformly over  $x \in U$ . Now  $R(x, y) = R(y, x)$ , so indeed  $R$  is smooth in  $\bar{U} \times \bar{U}$ .  $\square$

**$\tau$ -dependent single layer potential.** In [18] (see also Proposition 3.1 below) it is shown that for any  $\tau \in \mathbb{R}$  with  $|\tau| \geq 1$  and  $\tau^2 \notin \text{Spec}(-\Delta_{g_0})$ , and given  $\delta > 1/2$ , one has a bounded linear operator

$$G_\tau : L_\delta^2(T) \rightarrow H_\delta^2(T) \cap H_{-\delta,0}^1(T)$$

such that  $u = G_\tau f$  is the unique solution in  $H_{-\infty,0}^1(T)$  of the equation  $e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u = f$  in  $T$ , for any  $f \in L_\delta^2(T)$ .

If  $\tau$  is as above, we define another operator

$$\begin{aligned} K_\tau : L_c^2(T) &\rightarrow H_{\text{loc}}^2(T) \cap H_{\text{loc},0}^1(T), \\ K_\tau f &= e^{-\tau x_1} G_\tau(e^{\tau x_1} f). \end{aligned}$$

From the properties of  $G_\tau$ , it immediately follows that  $K_\tau$  is an inverse for the Laplacian: one has

$$-\Delta_g K_\tau = \text{Id} \quad \text{on } L_c^2(T).$$

The map  $K_\tau$  is a *Faddeev type Green operator* on  $T$ . It differs from the standard Green operator  $K_0$  by having exponential factors in its integral kernel. Also,  $K_\tau - K_0$  maps  $L_c^2(T)$  into  $C^\infty(T)$  since  $-\Delta_g(K_\tau - K_0)f = 0$  for all  $f \in L_c^2(T)$ . This suggests the following result. We will do the proof with some care to ensure that the boundary  $\partial T$  does not pose a problem.

**Lemma 2.4.**  $K_\tau = K_0 + R_\tau$  where  $R_\tau$  is an integral operator with kernel in  $C^\infty(T \times T)$ .

*Proof.* We prove the result for  $\tau \geq 1$ , the case  $\tau \leq -1$  being similar. The expression for  $G_\tau$  in [18, Proposition 4.1] shows that for any  $f \in L_\delta^2(T)$  with  $\delta > 1/2$  we have

$$G_\tau f(x_1, x') = - \sum_{l=1}^{\infty} [T_{\tau+\sqrt{\lambda_l}} T_{\tau-\sqrt{\lambda_l}} \tilde{f}(\cdot, l)](x_1) \phi_l(x')$$

with convergence in  $L_{-\delta}^2(T)$ , where for  $\mu \in \mathbb{R} \setminus \{0\}$  we define

$$T_\mu v(t) = \mathcal{F}_\eta^{-1} \left\{ \frac{1}{i\eta - \mu} \hat{v}(\eta) \right\} (t), \quad v \in \mathcal{S}'(\mathbb{R}).$$

This implies that for  $f \in L_c^2(T)$  one has

$$\begin{aligned} K_\tau f(x_1, x') &= - \sum_{l=1}^{\infty} e^{-\tau x_1} [T_{\tau+\sqrt{\lambda_l}} T_{\tau-\sqrt{\lambda_l}} (e^{\tau \cdot} \tilde{f}(\cdot, l))](x_1) \phi_l(x'), \\ K_0 f(x_1, x') &= - \sum_{l=1}^{\infty} [T_{\sqrt{\lambda_l}} T_{-\sqrt{\lambda_l}} \tilde{f}(\cdot, l)](x_1) \phi_l(x') \end{aligned}$$

with convergence in  $L_{\text{loc}}^2(T)$ . The second identity follows by solving the equation  $-\Delta_g u = f$  in  $T$ ,  $f \in L^2(T)$ ,  $u|_{\partial T} = 0$ , by taking the Fourier

transform in  $x_1$  and expanding in terms of Dirichlet eigenfunctions in  $x'$ . Note that  $K_0$  is obtained from  $K_\tau$  by setting  $\tau = 0$  formally.

We compute for  $v \in L_c^2(\mathbb{R})$

$$\begin{aligned} & -e^{-\tau x_1} [T_{\tau+\sqrt{\lambda_l}} T_{\tau-\sqrt{\lambda_l}} (e^{\tau \cdot} v)](x_1) + T_{\sqrt{\lambda_l}} T_{-\sqrt{\lambda_l}} v(x_1) \\ & = \int_{-\infty}^{\infty} [e^{-\tau(x_1-y_1)} m_\tau(x_1-y_1, \sqrt{\lambda_l}) - m_0(x_1-y_1, \sqrt{\lambda_l})] v(y_1) dy_1 \end{aligned}$$

where for  $a > 0$

$$m_\tau(t, a) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\eta} \frac{1}{(i\eta - (\tau + a))(i\eta - (\tau - a))} d\eta.$$

We claim that

$$e^{-\tau r} m_\tau(r, a) - m_0(r, a) = \begin{cases} 0, & \tau < a, \\ -\frac{1}{2a} e^{-ar}, & \tau > a. \end{cases} \quad (2.1)$$

If this holds, then using that  $\tau \neq \sqrt{\lambda_l}$  by the condition  $\tau^2 \notin \text{Spec}(-\Delta_{g_0})$  we obtain

$$\begin{aligned} (K_\tau - K_0)f(x_1, x') &= - \sum_{l; \sqrt{\lambda_l} < \tau} \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\lambda_l}} e^{-\sqrt{\lambda_l}(x_1-y_1)} \tilde{f}(y_1, l) \phi_l(x') dy_1 \\ &= \int_T R_\tau(x, y) f(y) dV(y) \end{aligned}$$

where  $R_\tau(x, y)$  is the integral kernel

$$R_\tau(x, y) = - \sum_{l; \sqrt{\lambda_l} < \tau} \frac{1}{2\sqrt{\lambda_l}} e^{-\sqrt{\lambda_l}(x_1-y_1)} \phi_l(x') \phi_l(y').$$

Since the sum is finite, the kernel is smooth in  $T \times T$  as required.

It remains to prove (2.1). Note that for  $r \in \mathbb{R}$  and  $a > 0$ ,

$$\begin{aligned} e^{-\tau r} m_\tau(r, a) &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(\eta+i\tau)r}}{((\eta+i\tau)i-a)((\eta+i\tau)i+a)} d\eta \\ &= \int_{\gamma_\tau} F(z) dz \end{aligned}$$

where the last expression is a contour integral over the curve  $\gamma_\tau(\eta) = \eta + i\tau$  for  $\eta \in (-\infty, \infty)$ , and

$$F(z) = -\frac{1}{2\pi} \frac{e^{izr}}{(iz-a)(iz+a)}.$$

It follows that

$$e^{-\tau r} m_\tau(r, a) - m_0(r, a) = \left( \int_{\gamma_\tau} - \int_{\gamma_0} \right) F(z) dz.$$

We may interpret the right hand side as a limit of integrals over closed rectangular contours since  $F(\pm R + it)$  where  $0 \leq t \leq \tau$  decays as  $R \rightarrow \infty$ . Now  $F(z)$  is analytic in  $\mathbb{C} \setminus \{\pm ia\}$  with simple poles at  $\pm ia$ , so (2.1) follows from the residue theorem.  $\square$

Since  $R_\tau u|_{\partial T} = 0$  for any  $u$ , the preceding result implies that

$$K_\tau : H_c^{-1}(T) \rightarrow H_{\text{loc},0}^1(T).$$

**Definition.** If  $|\tau| \geq 1$  and  $\tau^2 \notin \text{Spec}(-\Delta_{g_0})$ , we define the  $\tau$ -dependent single layer potential

$$S_\tau = K_\tau \gamma^* : H^{-1/2}(\Gamma) \rightarrow H_{\text{loc},0}^1(T).$$

The basic properties of  $S_\tau$  follow immediately from the previous results.

**Lemma 2.5.** Let  $|\tau| \geq 1$  and  $\tau^2 \notin \text{Spec}(-\Delta_{g_0})$ . If  $f \in H^{-1/2}(\Gamma)$  then  $u = S_\tau f \in H_{\text{loc},0}^1(T)$  satisfies

$$\begin{aligned} -\Delta_g u &= 0 \quad \text{in } M_\pm, \\ [u]_\Gamma &= 0, \\ [\partial_\nu u]_\Gamma &= f. \end{aligned}$$

If  $f \in H^s(\Gamma)$  for  $s \geq -1/2$  then  $u|_{M_-} \in H^{s+3/2}(M_-)$  and  $u|_{M_+} = \tilde{u}|_{M_+}$  for some  $\tilde{u} \in H_{\text{loc}}^{s+3/2}(T) \cap H_{\text{loc},0}^1(T)$ . The trace single layer potential satisfies

$$\gamma S_\tau : H^s(\Gamma) \rightarrow H^{s+1}(\Gamma), \quad s \geq -1/2.$$

One has

$$S_\tau f(x) = \int_\Gamma K_\tau(x, y) f(y) dS(y)$$

where the kernel  $K_\tau(x, y)$  is smooth off the diagonal in  $T^{\text{int}} \times T^{\text{int}}$  and  $\Delta_g(K_\tau(x, \cdot)) = 0$  in  $T^{\text{int}} \setminus \{x\}$ .

### 3. BOUNDARY INTEGRAL EQUATION

We now describe four equivalent problems for characterizing the CGO solutions  $u$ : a differential equation (DE), integral equation (IE), exterior problem (EP), and boundary integral equation (BE). In this section we assume that  $(M, g) \subset\subset (T^{\text{int}}, g)$  is a compact manifold with smooth boundary and  $T = \mathbb{R} \times M_0$ ,  $g = e \oplus g_0$ , and  $(M_0, g_0)$  is any compact  $(n-1)$ -dimensional manifold with smooth boundary. We use the notations in Section 2.

As a first step, we quote the basic existence and uniqueness result concerning  $G_\tau$  from [18, Proposition 4.3 and subsequent remark].

**Proposition 3.1.** Let  $\delta > 1/2$  and  $\lambda \neq 0$ , and suppose that  $q \in L^\infty(T)$  is compactly supported. There exists  $\tau_0 \geq 1$  (if  $q = 0$  then  $\tau_0 = 1$ ) such that whenever

$$|\tau| \geq \tau_0 \quad \text{and} \quad \tau^2 \notin \text{Spec}(-\Delta_{g_0}),$$

then for any  $f = f_1 + f_2$  where  $f_1 \in L_\delta^2(T)$ ,  $f_2 \in L_{-\delta}^2(T)$ , and

$$\mathcal{F}_{x_1} f_2(\cdot, x') \text{ has support in } \{|\xi_1| \geq |\lambda|\} \text{ for a.e. } x' \in M_0,$$

there is a unique solution  $w \in H_{-\infty,0}^1(T)$  of the equation

$$e^{\tau x_1}(-\Delta + q)e^{-\tau x_1}w = f \quad \text{in } T.$$

Further,  $w \in H_{-\delta,0}^1(T) \cap H_{-\delta}^2(T)$ , and one has  $w = G_\tau v$  where

$$(\text{Id} + qG_\tau)v = f, \quad \|(\text{Id} + qG_\tau)^{-1}\|_{L_\delta^2 \rightarrow L_\delta^2} \leq 2.$$

Finally,  $w$  satisfies the estimates

$$\|w\|_{H_{-\delta}^s(T)} \leq C|\tau|^{s-1} \left[ \|f_1\|_{L_\delta^2(T)} + \|f_2\|_{L_{-\delta}^2(T)} \right], \quad 0 \leq s \leq 2,$$

with  $C$  independent of  $\tau$  and  $f_1, f_2$ .

In the following result we extend  $q \in L^\infty(M)$  by zero into  $T$ , and let  $\tau_0$  be as in Proposition 3.1. The result considers CGO solutions of the form

$$u = u_0 + e^{-\tau x_1} r$$

where  $u_0$  is any harmonic function in  $H_{\text{loc}}^2(T)$ , and  $r \in H_{-\infty,0}^1(T)$ . We will only fix the choice of the free solution  $u_0$  in the next section.

**Proposition 3.2.** Let  $q \in L^\infty(M)$  be such that 0 is not a Dirichlet eigenvalue of  $-\Delta_g + q$  in  $M$ , and let  $|\tau| \geq \tau_0$  and  $\tau^2 \notin \text{Spec}(-\Delta_{g_0})$ . Further, let  $u_0 \in H_{\text{loc}}^2(T)$  be such that  $\Delta_g u_0 = 0$  in  $T$ . Consider the following problems:

$$\begin{aligned} \text{(DE)} & \begin{cases} (-\Delta_g + q)u = 0 \text{ in } T \\ e^{\tau x_1}(u - u_0) \in H_{-\infty,0}^1(T), \end{cases} \\ \text{(IE)} & \begin{cases} u + K_\tau(qu) = u_0 \text{ in } T \\ u \in H_{\text{loc}}^2(T), \end{cases} \\ \text{(EP)} & \begin{cases} \text{i) } \Delta_g u = 0 \text{ in } T \setminus M \\ \text{ii) } u = \tilde{u}|_{T \setminus M} \text{ for some } \tilde{u} \in H_{\text{loc}}^2(T) \\ \text{iii) } e^{\tau x_1}(u - u_0)|_{T \setminus M} = \tilde{r}|_{T \setminus M} \text{ for some } \tilde{r} \in H_{-\infty,0}^1(T) \\ \text{iv) } (\partial_\nu u)_+ = \Lambda_{g,q}(\gamma + u) \text{ on } \Gamma, \end{cases} \\ \text{(BE)} & \begin{cases} (\text{Id} + \gamma S_\tau(\Lambda_{g,q} - \Lambda_{g,0}))f = u_0 \text{ on } \Gamma \\ f \in H^{3/2}(\Gamma). \end{cases} \end{aligned}$$

Each of these problems has a unique solution. Further, these problems are equivalent in the sense that  $u$  solves (DE) iff  $u$  solves (IE), if  $u$  solves (DE) then  $u|_{T \setminus M}$  solves (EP), if  $u$  solves (EP) then there is a solution  $\tilde{u}$  of (DE) with  $\tilde{u}|_{T \setminus M} = u$ , if  $u$  solves (DE) then  $f = u|_\Gamma$  solves (BE), and finally if  $f$  solves (BE) then there is a solution  $u$  of (DE) with  $u|_\Gamma = f$ .

*Proof.* The function  $u = u_0 + e^{-\tau x_1} r$  solves  $(-\Delta_g + q)u = 0$  in  $T$  if and only if

$$e^{\tau x_1}(-\Delta_g + q)e^{-\tau x_1} r = -e^{\tau x_1} q u_0 \quad \text{in } T.$$

The right hand side is in  $L_c^2(T)$ , so by Proposition 3.1 there is a unique solution  $r \in H_{-\infty,0}^1(T)$ . This proves that (DE) has a unique solution. It remains to prove that all four problems are equivalent in the sense described above.

(DE)  $\implies$  (IE): Assume  $u$  solves (DE). Then  $u = u_0 + e^{-\tau x_1} r$  where  $r \in H_{-\infty,0}^1(T)$ , and

$$e^{\tau x_1}(-\Delta_g + q)e^{-\tau x_1} r = -e^{\tau x_1} q u_0 \quad \text{in } T.$$

By Proposition 3.1 we have  $r = G_\tau v \in H_{\text{loc}}^2(T)$  where  $v$  satisfies

$$v + qr = -e^{\tau x_1} q u_0.$$

Since  $q$  is compactly supported in  $T$  also  $v = -q(r + e^{\tau x_1} u_0)$  is compactly supported. Thus we may apply  $G_\tau$  to both sides of the last identity to obtain

$$r + G_\tau(qr) = -G_\tau(e^{\tau x_1} q u_0).$$

Multiplying by  $e^{-\tau x_1}$  and adding  $u_0$  to both sides gives (IE).

(IE)  $\implies$  (DE): Assume  $u$  solves (IE). Then the function  $r = e^{\tau x_1}(u - u_0)$  satisfies

$$r = -G_\tau(e^{\tau x_1}qu). \quad (3.1)$$

This shows that  $r \in H_{-\infty,0}^1(T)$ , and (DE) follows by applying  $-\Delta_g$  to both sides of (IE).

(DE)  $\implies$  (EP): Let  $\tilde{u}$  solve (DE), and define  $u = \tilde{u}|_{T \setminus M}$ . Clearly properties i), ii) and iii) of (EP) are valid. We need to show iv). Since  $\tilde{u}$  solves the equation  $(-\Delta_g + q)\tilde{u} = 0$  in  $M$ , we have

$$(\partial_\nu u)_+ = \partial_\nu \tilde{u}|_\Gamma = \Lambda_{g,q}(\tilde{u}|_\Gamma) = \Lambda_{g,q}(\gamma_+ u).$$

(EP)  $\implies$  (DE): Suppose  $u$  solves (EP). Define  $v \in H^2(M)$  as the unique solution of the equation  $(-\Delta_g + q)v = 0$  in  $M$  with  $v|_\Gamma = \gamma_+ u|_\Gamma$ , and define

$$\tilde{u}(x) = \begin{cases} v(x), & x \in M, \\ u(x), & x \in T \setminus M. \end{cases}$$

Then  $\gamma_- \tilde{u}|_\Gamma = \gamma_+ \tilde{u}|_\Gamma$  and

$$(\partial_\nu \tilde{u})_-|_\Gamma = \Lambda_{g,q}(\gamma_+ u|_\Gamma) = (\partial_\nu \tilde{u})_+|_\Gamma$$

by (EP) iv). It follows that  $\tilde{u} \in H_{\text{loc}}^2(T)$  and  $(-\Delta_g + q)\tilde{u} = 0$  in  $T$ . Further,  $e^{\tau x_1}(\tilde{u} - u_0) \in H_{-\infty,0}^1(T)$  by (EP) iii).

(DE)  $\implies$  (BE): Let  $u$  solve (DE), and let  $f = u|_\Gamma$ . We fix a point  $x \in T^{\text{int}} \setminus M$  and let  $v(y) = K_\tau(x, y)$  where  $y \in M$ . This is a smooth function in  $M$  by Lemma 2.5.

Now Green's theorem implies

$$\int_\Gamma (u \partial_\nu v - v \partial_\nu u) dS = \int_M (u \Delta_g v - v \Delta_g u) dV.$$

By (DE) we have  $\Delta_g u = qu$  and  $\partial_\nu u|_\Gamma = \Lambda_{g,q}f$ . Using the properties in Lemma 2.5 we obtain

$$\int_\Gamma u \partial_\nu v dS - S_\tau \Lambda_{g,q}f(x) = -K_\tau(qu)(x),$$

which is valid for  $x \in T^{\text{int}} \setminus M$ . The function  $v$  is harmonic in  $M$ , hence  $\partial_\nu v|_\Gamma = \Lambda_{g,0}(v|_\Gamma)$ . The symmetry of  $\Lambda_{g,0}$  implies

$$\int_\Gamma u \partial_\nu v dS = \int_\Gamma u \Lambda_{g,0}(v|_\Gamma) dS = \int_\Gamma \Lambda_{g,0}(u|_\Gamma) v dS = S_\tau \Lambda_{g,0}f(x).$$

We obtain

$$S_\tau(\Lambda_{g,q} - \Lambda_{g,0})f = K_\tau(qu) \quad \text{in } T^{\text{int}} \setminus M. \quad (3.2)$$

Adding  $u$  to both sides, using the fact that  $u$  solves (IE), and taking traces on  $\Gamma$  gives (BE).

(BE)  $\implies$  (EP): Let  $f$  solve (BE). We define a function  $\tilde{u} \in H_{\text{loc}}^1(T)$  by

$$\tilde{u} = u_0 - S_\tau(\Lambda_{g,q} - \Lambda_{g,0})f.$$

This function is harmonic in  $T \setminus \Gamma$  by Lemma 2.5, and  $\tilde{u}|_\Gamma = f$  by using (BE). The jump relation for  $S_\tau$  implies that on  $\Gamma$

$$(\partial_\nu \tilde{u})_- - (\partial_\nu \tilde{u})_+ = -(\Lambda_{g,q} - \Lambda_{g,0})f.$$

But  $(\partial_\nu \tilde{u})_- = \Lambda_{g,0}f$ , so we have  $(\partial_\nu \tilde{u})_+ = \Lambda_{g,q}(\gamma_+ \tilde{u})$ . Therefore  $\tilde{u}|_{T \setminus M}$  satisfies (EP) i) and iv). Also (EP) ii) is valid by mapping properties of  $S_\tau$ .

To prove (EP) iii) it is sufficient to show that for any  $h \in H^{1/2}(\Gamma)$ ,

$$e^{\tau x_1} S_\tau h|_{T \setminus M} = w|_{T \setminus M} \quad \text{for some } w \in H_{-\infty,0}^1(T). \quad (3.3)$$

Formally one has  $e^{\tau x_1} S_\tau h = G_\tau e^{\tau x_1} \gamma^* h$  where  $G_\tau$  maps  $L_c^2(T)$  to  $H_{-\infty,0}^1(T)$ . However, we have not proved that  $G_\tau$  has good mapping properties on negative order Sobolev spaces. Thus, the proof proceeds differently and involves an extension  $w$  of  $e^{\tau x_1} S_\tau h|_{T \setminus M}$  into  $T$  such that  $w = G_\tau \psi$  for some  $\psi \in L_c^2(T)$ . This will imply (3.3) by the mapping properties of  $G_\tau$ .

Define

$$w(x) = \begin{cases} e^{\tau x_1} (P_0(\gamma S_\tau h) + F), & x \in M, \\ e^{\tau x_1} S_\tau h, & x \in T \setminus M \end{cases}$$

where  $P_0 : H^{3/2}(\Gamma) \rightarrow H^2(M)$  is the Poisson operator mapping  $h_0$  to the function  $v_0$  with  $-\Delta_g v_0 = 0$  in  $M$  and  $v_0|_\Gamma = h_0$ , and  $F \in H^2(M)$  is a function chosen so that  $e^{-\tau x_1} w \in H_{\text{loc}}^2(T)$ . Clearly we need that  $F|_\Gamma = 0$ , and since

$$[\partial_\nu(e^{-\tau x_1} w)]_\Gamma = \Lambda_{g,0}(\gamma S_\tau h) + \partial_\nu F|_\Gamma - (\partial_\nu S_\tau h)_+$$

we also require that  $\partial_\nu F|_\Gamma = (\partial_\nu S_\tau h)_+ - \Lambda_{g,0}(\gamma S_\tau h) \in H^{1/2}(\Gamma)$ . We can take  $F$  to be any function in  $H^2(M)$  with this Cauchy data, and then  $e^{-\tau x_1} w$  and also  $w$  is in  $H_{\text{loc}}^2(T)$ .

We now observe that

$$e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} w(x) = \begin{cases} -e^{\tau x_1} \Delta_g F, & x \in M, \\ 0, & x \in T \setminus M. \end{cases}$$

Since  $w \in H_{\text{loc}}^2(T)$  this implies that  $e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} w = \psi$  where  $\psi \in L_c^2(T)$ . Consequently  $w = G_\tau \psi \in H_{-\infty,0}^1(T)$  and we have proved (3.3).  $\square$

Finally, let us verify that the boundary integral equation (BE) in Proposition 3.2 is indeed Fredholm.

**Proposition 3.3.** The operator

$$\gamma S_\tau (\Lambda_{g,q} - \Lambda_{g,0}) : H^{3/2}(\Gamma) \rightarrow H^{3/2}(\Gamma)$$

is compact.

*Proof.* Let  $f \in H^{3/2}(\Gamma)$ , and let  $u = P_q f$  where  $P_q : H^{3/2}(\Gamma) \rightarrow H^2(M)$  is the Poisson operator mapping  $h_0$  to  $v_0$  where  $(-\Delta_g + q)v_0 = 0$  in  $M$  and  $v_0|_\Gamma = h_0$ . The exact same argument leading to (3.2) in the proof of Proposition 3.2 shows that

$$S_\tau (\Lambda_{g,q} - \Lambda_{g,0}) f = K_\tau (q E J u) \quad \text{in } T^{\text{int}} \setminus M$$

where  $E : L^2(M) \rightarrow L^2(T)$  is extension by zero and  $J : H^2(M) \rightarrow L^2(M)$  is the natural inclusion. Taking traces on  $\Gamma$ , we obtain the factorization

$$\gamma S_\tau (\Lambda_{g,q} - \Lambda_{g,0}) = \gamma K_\tau q E J P_q.$$

The result follows since on the right hand side  $J$  is compact and all other operators are bounded.  $\square$

## 4. PROOFS OF THE MAIN RESULTS

In Sections 2 and 3 we considered layer potentials and equivalent problems characterizing CGO solutions in the case where  $(M, g) \subset\subset (T^{\text{int}}, g)$  where  $T = \mathbb{R} \times M_0$ ,  $g = e \oplus g_0$ , and  $(M_0, g_0)$  can be any compact  $(n-1)$ -dimensional manifold with boundary. Now we specialize to the case where  $(M_0, g_0)$  is simple and prove Theorems 1.1 and 1.2.

The first step is to fix the harmonic functions  $u_0$  used in Proposition 3.2. We first choose a simple manifold  $(\tilde{M}_0, g_0)$  such that  $(M_0, g_0) \subset\subset (\tilde{M}_0, g_0)$ . Below, we will write  $(r, \theta)$  for the polar normal coordinates in  $(\tilde{M}_0, g_0)$  with center at a given point  $p \in \tilde{M}_0 \setminus M_0$  (these exist globally because the manifold is simple), and we write, following [7, Section 5],

$$\tilde{a} = \tilde{a}(x_1, r, \theta) = e^{-i\tau r} |g|^{-1/4} e^{i\lambda(x_1 + ir)} b(\theta)$$

where  $\lambda$  is a fixed nonzero real number and  $b \in C^\infty(S^{n-2})$  is a fixed function. Note that  $\tilde{a} \in C^\infty(T)$  since the coordinates  $(r, \theta)$  are smooth in  $M_0$ .

**Proposition 4.1.** Given any  $\tau$  with  $|\tau| \geq 1$  and  $\tau^2 \notin \text{Spec}(-\Delta_{g_0})$ , and for any point  $p \in \tilde{M}_0 \setminus M_0$ , for any real number  $\lambda \neq 0$ , and for any smooth function  $b = b(\theta)$ , there is a function  $u_0$  with

$$\Delta_g u_0 = 0 \quad \text{in } T, \quad u_0 \in H_{\text{loc}}^2(T),$$

of the form

$$u_0 = e^{-\tau x_1} \tilde{a} + e^{-\tau x_1} r_0$$

where  $r_0 = G_\tau f$  for some explicit function  $f$  and  $\|r_0\|_{L^2(M)} = O(|\tau|^{-1})$  as  $|\tau| \rightarrow \infty$ .

*Proof.* If  $u_0$  is of the required form, then  $\Delta_g u_0 = 0$  is equivalent with

$$e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} r_0 = f \tag{4.1}$$

where  $f = e^{\tau x_1} \Delta_g (e^{-\tau x_1} \tilde{a})$ . Writing  $\Phi = x_1 + ir$ , we compute

$$\begin{aligned} f &= -e^{-i\tau r} [e^{\tau\Phi} (-\Delta_g) e^{-\tau\Phi}] (|g|^{-1/4} e^{i\lambda(x_1 + ir)} b(\theta)) \\ &= -e^{-i\tau r} [-\tau^2 \langle d\Phi, d\Phi \rangle + \tau(2\langle d\Phi, d\cdot \rangle + \Delta_g \Phi) - \Delta_g] (|g|^{-1/4} e^{i\lambda(x_1 + ir)} b(\theta)). \end{aligned}$$

Here we have extended  $\langle \cdot, \cdot \rangle$  as a complex bilinear form to complex valued 1-forms. As in [7, Section 5], we see that  $\langle d\Phi, d\Phi \rangle = 0$  and also that  $(2\langle d\Phi, d\cdot \rangle + \Delta_g \Phi)(|g|^{-1/4} e^{i\lambda(x_1 + ir)} b(\theta)) = 0$  (this was the reason for the choice of  $\tilde{a}$ ). Consequently

$$\begin{aligned} f &= e^{-i\tau r} \Delta_g (e^{i\tau r} \tilde{a}) = e^{-i\tau r} (\partial_1^2 + \Delta_{g_0}) (e^{i\tau r} \tilde{a}) \\ &= e^{i\lambda x_1} [e^{-i\tau r} (\Delta_{g_0} - \lambda^2) (|g|^{-1/4} e^{-\lambda r} b(\theta))]. \end{aligned}$$

Then for any  $\delta > 1/2$  one has  $f \in L_{-\delta}^2(T)$ , the norm  $\|f\|_{L_{-\delta}^2(T)}$  is independent of  $\tau$ , and the Fourier transform  $\mathcal{F}_{x_1} f(\cdot, x')$  is supported in  $\{|\xi_1| \geq |\lambda|\}$ . By Proposition 3.1 we have a solution  $r_0 = G_\tau f$  of (4.1), which gives the required solution  $u_0$ .  $\square$

We can now prove the main theorems.

*Proof of Theorem 1.2.* We first consider the case, as in the beginning of this section, where  $(M, g)$  is an admissible manifold with conformal factor  $c = 1$ . Suppose that the manifold  $(M, g)$ , and consequently also  $(M_0, g_0)$ , and the map  $\Lambda_{g,q}$  are known. We wish to determine  $q$  from this knowledge.

First note the basic integral identity (see [7, Lemma 6.1])

$$\int_{\partial M} ((\Lambda_{g,q} - \Lambda_{g,0})f_1)f_2 dS = \int_M qu_1u_2 dV \quad (4.2)$$

which is valid for any  $u_j \in H^2(M)$  with  $(-\Delta + q)u_1 = 0$  in  $M$ ,  $\Delta_g u_2 = 0$  in  $M$ , and  $u_j|_{\partial M} = f_j$ . We consider CGO solutions in  $T$  of the form

$$\begin{aligned} u_1 &= u_{0,1} + e^{-\tau x_1} r_1, \\ u_2 &= u_{0,2} \end{aligned}$$

where  $u_{0,j}$  are harmonic functions provided by Proposition 4.1 having the form

$$\begin{aligned} u_{0,1} &= e^{-\tau(x_1+ir)} |g|^{-1/4} e^{i\lambda(x_1+ir)} b(\theta) + e^{-\tau x_1} G_\tau \psi_1, \\ u_{0,2} &= e^{\tau(x_1+ir)} |g|^{-1/4} e^{i\lambda(x_1+ir)} + e^{\tau x_1} G_{-\tau} \psi_2. \end{aligned}$$

Here  $\tau \geq \tau_0$  and  $\tau^2 \notin \text{Spec}(-\Delta_{g_0})$ ,  $(r, \theta)$  are polar normal coordinates in  $(\tilde{M}_0, g_0)$  with center at  $p \in \tilde{M}_0 \setminus M_0$ ,  $\lambda \neq 0$ ,  $b$  is a smooth function in  $S^{n-2}$ , and  $\psi_j$  are explicit functions with  $\|G_{\pm\tau}\psi_j\|_{L^2(M)} = O(|\tau|^{-1})$ .

The point is that  $u_{0,j}$  are explicit functions which can be constructed from the knowledge of  $(M, g)$ , and also  $f_2 = u_{0,2}|_\Gamma$  is known. By Proposition 3.2 there is a unique CGO solution  $u_1$  of the above form, and the boundary value  $f_1 = u_1|_\Gamma$  is the unique solution in  $H^{3/2}(\Gamma)$  of the boundary integral equation

$$(\text{Id} + \gamma S_\tau(\Lambda_{g,q} - \Lambda_{g,0}))f_1 = u_{0,1} \quad \text{on } \Gamma.$$

Since the operator on the left and the function on the right are known from our data, we can construct  $f_1$  as the unique solution of this Fredholm integral equation. Then the left hand side of (4.2) is known, and consequently we can determine from our data the integrals

$$\int_M qu_1u_2 dV \quad (4.3)$$

for any  $u_1$  and  $u_2$  as above.

Since  $u_1$  solves  $(-\Delta_g + q)u_1 = 0$  with  $r_1 \in H^1_{-\infty,0}(T)$ , Proposition 3.1 shows that

$$r_1 = -G_\tau(\text{Id} + qG_\tau)^{-1}(e^{\tau x_1} q u_{0,1})$$

and using the form of  $u_{0,1}$  and norm estimates for  $G_\tau$  gives

$$\|r_1\|_{L^2(M)} = O(|\tau|^{-1}).$$

Thus, taking the limit as  $\tau \rightarrow \infty$  in (4.3), we have recovered from our boundary data the quantities

$$\int_M q|g|^{-1/2} e^{2i\lambda(x_1+ir)} b(\theta) dV.$$

At this point it is convenient to extend  $q$  into  $T$  as a function in  $C_c^\infty(T^{\text{int}})$ . This may be done by recovering the Taylor series of  $q$  on  $\partial M$  via boundary determination [7, Section 8] (this procedure is constructive), and by extending  $q$  to a function in  $C_c^\infty(T^{\text{int}})$  so that  $q|_{T \setminus M}$  is known. Using that  $dV = |g|^{1/2} dx_1 dr d\theta$ , the last integral becomes

$$\int_0^\infty \int_{S^{n-2}} e^{-2\lambda r} \left[ \int_{-\infty}^\infty e^{2i\lambda x_1} q(x_1, r, \theta) dx_1 \right] b(\theta) dr d\theta.$$

Denoting the quantity in brackets by  $f_\lambda(r, \theta)$  and by varying the smooth function  $b$ , we determine the integrals

$$\int_0^\infty e^{-2\lambda r} f_\lambda(r, \theta) dr \quad \text{for all } \theta \in S^{n-2}.$$

These integrals are known for any nonzero real number  $\lambda$  and for any point  $p \in \tilde{M}_0 \setminus M_0$  which is the center of the polar normal coordinates  $(r, \theta)$  in  $\tilde{M}_0$ . Noting that  $r \mapsto (r, \theta)$  is the unit speed geodesic in  $(\tilde{M}_0, g_0)$  starting at  $p$  in direction  $\theta$ , and letting  $p$  approach  $\partial M_0$ , we can recover the integrals

$$\int_0^T e^{-2\lambda r} f_\lambda(\gamma(r)) dr \quad (4.4)$$

for any geodesic  $\gamma : [0, T] \rightarrow M_0$  where  $\gamma(0), \gamma(T) \in \partial M_0$  and  $\gamma(t)$  for  $0 < t < T$  lies in  $M_0^{\text{int}}$ . This is the attenuated geodesic ray transform of  $f_\lambda$  in  $(M_0, g_0)$ , with constant attenuation  $-2\lambda$ . Now, assuming  $\dim(M) = 3$  so  $(M_0, g_0)$  is 2-dimensional, we invoke the invertibility result for the attenuated ray transform [28] which allows to recover the function  $f_\lambda$  in  $M_0$  from the integrals (4.4) for any  $\lambda$ . Thus, we have determined the integrals

$$\int_{-\infty}^\infty e^{2i\lambda x_1} q(x_1, x') dx_1$$

for any  $\lambda \neq 0$  and for any  $x' \in M_0$ . This determines  $q$  in  $M$  by inverting the one-dimensional Fourier transform.

We have proved the theorem in the case where  $(M, g)$  is an admissible manifold and with conformal factor  $c = 1$ . For general conformal factors, suppose that  $(M, g)$  is admissible and  $g = c\tilde{g}$  where  $\tilde{g} = e \oplus g_0$ . Define also  $\tilde{q} = c(q - q_c)$  where  $q_c = c^{\frac{n-2}{4}} \Delta_{c\tilde{g}}(c^{-\frac{n-2}{4}})$ . The identity

$$c^{\frac{n+2}{4}} (-\Delta_{c\tilde{g}} + q)(c^{-\frac{n-2}{4}} u) = (-\Delta_{\tilde{g}} + \tilde{q})u$$

implies that, since  $\nu_{\tilde{g}} = c^{1/2}\nu_g$ ,

$$\Lambda_{\tilde{g}, \tilde{q}} f = c^{\frac{n}{4}} \Lambda_{g, q}(c^{-\frac{n-2}{4}} f) + \frac{n-2}{4} c^{-1} (\partial_{\nu_{\tilde{g}}} c) f. \quad (4.5)$$

Thus, from the knowledge of  $\Lambda_{g, q}$  and  $(M, g)$  we can determine  $\Lambda_{\tilde{g}, \tilde{q}}$ . The proof above then shows that one can reconstruct  $\tilde{q}$ , from which  $q$  is easily determined.  $\square$

*Proof of Theorem 1.1.* Let  $(M, \tilde{g})$  be admissible and known and suppose that  $\Lambda_{c\tilde{g}} = \Lambda_{c\tilde{g}, 0}$  is known. By boundary determination [7] we can determine  $c|_{\partial M}$  and  $\partial_{\nu_{\tilde{g}}} c|_{\partial M}$ . The identity (4.5) shows that

$$\Lambda_{\tilde{g}, \tilde{q}} f = c^{\frac{n}{4}} \Lambda_{c\tilde{g}, 0}(c^{-\frac{n-2}{4}} f) + \frac{n-2}{4} c^{-1} (\partial_{\nu_{\tilde{g}}} c) f$$

with  $\tilde{q} = -c^{\frac{n+2}{4}} \Delta_{c\tilde{g}}(c^{-\frac{n-2}{4}})$ . This shows that  $\Lambda_{c\tilde{g}}$  determines  $\Lambda_{\tilde{g},\tilde{q}}$ , and Theorem 1.2 implies that we can recover  $\tilde{q}$ .

We write  $w = \log c^{-\frac{n-2}{4}}$  and compute

$$\begin{aligned} \Delta_{\tilde{g}}w &= \sum_{j,k=1}^n |\tilde{g}|^{-1/2} \partial_j (|\tilde{g}|^{1/2} \tilde{g}^{jk} c^{\frac{n-2}{4}} \partial_k (c^{-\frac{n-2}{4}})) \\ &= \sum_{j,k=1}^n c^{n/2} |c\tilde{g}|^{-1/2} \partial_j (c^{-\frac{n-2}{4}} |c\tilde{g}|^{1/2} (c\tilde{g})^{jk} \partial_k (c^{-\frac{n-2}{4}})) \\ &= c^{\frac{n+2}{4}} \Delta_{c\tilde{g}}(c^{-\frac{n-2}{4}}) + \sum_{j,k=1}^n c^{\frac{n-2}{2}} \tilde{g}^{jk} \partial_j (c^{-\frac{n-2}{4}}) \partial_k (c^{-\frac{n-2}{4}}). \end{aligned}$$

This implies that

$$\begin{aligned} -\Delta_{\tilde{g}}w + \langle dw, dw \rangle_{\tilde{g}} &= \tilde{q} \quad \text{in } M, \\ w|_{\partial M} &= \log c^{-\frac{n-2}{4}}|_{\partial M}. \end{aligned}$$

This nonlinear Dirichlet problem has a unique solution by the maximum principle [9] and we have already recovered the right hand side  $\tilde{q}$  and the boundary value  $\log c^{-\frac{n-2}{4}}|_{\partial M}$ , so we may construct  $w$  in  $M$  by solving the problem. This determines  $c$  in  $M$ .  $\square$

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