Focusing Waves in Elctromagnetic Inverse Problems

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1 Introduction

This paper is devoted to the inverse boundary-value problem of electromagnetics in the case $\sigma = 0$, where σ is the conductivity. Thus the governing Maxwell equations, in the time domain, are of the form

$$\operatorname{curl} E(x,t) = -B_t(x,t), \ \text{(Maxwell–Faraday)}, \qquad \operatorname{div} B(x,t) = 0, \tag{1}$$

$$\operatorname{curl} H(x,t) = \quad D_t(x,t), \ \text{(Maxwell–Ampère)}, \quad \operatorname{div} D(x,t) = 0, \tag{2}$$

$(x,t) \in \mathcal{N} \times \mathbb{R}$, $\mathcal{N} \subset \mathbb{R}^3$ - a bounded domain, together with the constitutive relations

$$D(x,t) = \epsilon(x)E(x,t), \quad B(x,t) = \mu(x)H(x,t). \tag{3}$$

Here ε, μ are 3×3 time-independent positive matrices, which, as also $\partial\mathcal{N}$, are C^∞-smooth.

The main results in the study of inverse boundary problems of electrodynamics deal with the isotropic case, i.e. scalar ε, μ. It is shown in [5], [15], [16] that the *static admittance map*,

$$\mathcal{Z}_0 : n \times E_0|_{\partial\mathcal{N}} \to n \times H_0|_{\partial\mathcal{N}},$$

where $(E_0,\, H_0)$ are stationary solutions to (1) - (3), determine ε, μ uniquely. What is more, the results of [5], [15], [16] make possible to find all three scalar coefficients, including conductivity, $\sigma \neq 0$.

There are some other approaches to the inverse problem for (1) – (3), working directly in the time-domain, [1], [17]. They make possible, under additional

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geometrical restrictions, to find some combinations of unknown parameters. Namely, [17] deals with the case when $\mathcal{N}$ is a simple geodesic manifold in the metric $dl^2 = \varepsilon\mu|dx|^2$, while constructions of [1] are valid in a collar neighborhood of $\partial\mathcal{N}$.

Much less is known in the anisotropic case. It is, however, clear from scalar anisotropic problems that, instead of the uniqueness, one obtains a group of transformations, involving proper coordinate changes in $\mathcal{N}$, e.g. [14], [19], [4], [8], [13]. Therefore, it is natural to split the solution of an anisotropic inverse problem into two steps. Firstly, to formulate and solve the corresponding coordinate-invariant inverse problem, i.e. an inverse problem on a manifold. Secondly, to analyse the properties of an inverse problem in $\mathbb{R}^n$ resulting from embedding the manifold into $\mathbb{R}^n$. (For a systematic development of this approach see [7]).

In this paper, we confine our study to the case of the *scalar wave impedance*,

$$\mu = \alpha^2\epsilon, \tag{4}$$

where α is a positive scalar function.

Let g be the metric on $\mathcal{N}$,

$$g^{ij} = \frac{1}{\alpha^2 \det(\varepsilon)} g_0^{ik}\varepsilon_k^j = \frac{\alpha^2}{\det(\mu)} g_0^{ik}\mu_k^j, \tag{5}$$

$g_0^{ij} = \delta^{ij}$, where the last equation in (5) is due to (4). Introduce differential 1− and 2− forms, $\omega^1 \in \Omega^1 M$, $\omega^2 \in \Omega^2 M$,

$$\omega^1 = E^\flat, \quad \omega^2 = *_0 B^\flat. \tag{6}$$

Here $*_0$ is the duality between 1− forms and vector fields,

$$X^\flat(Y) = g_0(X,Y),$$

X, Y being arbitrary vector fields, and $*_0$ is the Hodge star-operator in metric g_0. Metric g_0 appears in these equations as a background metric which, in case of $\mathcal{N} \subset \mathbb{R}^3$ is the canonical Euclidian metric. However, we can assume from the beginning that $(\mathcal{N}, g_0)$ is a compact 3−dimensional Riemannian manifold with invariantly defined , in metric g_0, operators curl and div in (1), (2). All further constructions remain valid in this general case.

Then equations (1) – (3) may be written as,

$$\omega_t^1 = \delta_\alpha\omega^2, \quad \delta_\alpha\omega^1 = 0, \tag{7}$$

$$\omega_t^2 = \ d\omega^1, \qquad d\omega^2 = 0, \tag{8}$$

with α−codifferential $\delta_\alpha : \Omega^k M \to \Omega^{3-k} M$ given by

$$\delta_\alpha\omega^k = (-1)^k \alpha * d\frac{1}{\alpha} * \omega^k, \tag{9}$$

and $*$ being the Hodge star-operator in metric g.

We note that constitutive relations (3) are now incorporated into (7), (8) via the new metric g which, for the reasons clear from the following, is called the *travel time metric*. We note that Maxwell equations, (1) – (3) in the form (7), (8), may also be written for another pair of differential forms

$$\eta^1 = \alpha H^\flat, \quad \eta^2 = *_0 \alpha D^\flat,$$

with the connection between two representations given by

$$\eta^2 = *\omega^1, \quad \eta^1 = *\omega^2. \tag{10}$$

This reflects the well-known duality of the Maxwell equations.

From now on M is a compact $3-$manifold with (travel time) metric g and wave impedance α, (M, g, α). To have an initial-boundary value problem, we compliment (7), (8) with initial and boundary conditions,

$$\omega^1|_{t=0} = 0, \quad \omega^2|_{t=0} = 0. \tag{11}$$

$$\mathbf{t}\omega^1 = f \in C_0^\infty(\mathbb{R}_+, \Omega^1 \partial M). \tag{12}$$

Here, $\mathbf{t}\omega^k$ is the tangential component of ω^k on ∂M, $\mathbf{t} : \Omega^k M \to \Omega^k \partial M$,

$$\mathbf{t}\omega^k = i^* \omega^k, \quad i : \partial M \to M. \tag{13}$$

To state rigorously the initial-boundary value problem (7), (8), (12), (13), the notion of the *complete Maxwell system*,

$$\omega_t + \mathcal{M}\omega = 0, \quad \omega = (\omega^0, \omega^1, \omega^2, \omega^3) \in \mathbf{\Omega} M, \tag{14}$$

is introduced.

Here $\mathbf{\Omega} M = \Omega^0 M \times \Omega^1 M \times \Omega^2 M \times \Omega^3 M$ is the full Grassmanian bundle over M and

$$\mathcal{M} = d - \delta_\alpha \tag{15}$$

is the Dirac-type operator on $\mathbf{\Omega} M$. Equations (14) are supplemented by initial and boundary conditions,

$$\omega|_{t=0} = 0, \quad \mathbf{t}\omega = f \in C_0^\infty(\mathbb{R}_+, \mathbf{\Omega}\partial M), \tag{16}$$

which give rise to a well-posed initial–boundary value problem. It turns out that the problem (14), (16) is equivalent to (7), (8), (11), (12), if $\mathbf{t}\omega^0 = 0$, $\partial_t \mathbf{t}\omega^2 = -d\mathbf{t}\omega^1$ (for details of the construction see [10], [11]).

Denote by $\mathcal{Z}_T$ the *admittance map*, $\mathcal{Z}_T : \overset{\circ}{C}{}^\infty([0,T], \Omega^1 \partial M) \to \overset{\circ}{C}{}^\infty([0,T], \Omega^1 \partial M)$,

$$\mathcal{Z}_T(f) = \mathbf{n}(\omega^f)^2(t), \quad t \in [0,T], \tag{17}$$

where $\omega^f(t) = \big((\omega^f)^1(t), (\omega^f)^2(t)\big)$ is the solution to (7), (8), (12), (13), $\mathbf{n}\omega^k$ is the normal component of ω^k on ∂M, $\mathbf{n} : \Omega^k M \to \Omega^{3-k}\partial M$

$$\mathbf{n}\omega^k = i^*(\frac{1}{\alpha} * \omega^k), \tag{18}$$

and $\overset{\circ}{C}{}^\infty$ consists of smooth functions vanishing near $t = 0$.

We are in the position now to formulate the main results of the paper.

Theorem 1.1 *Let $T > 2rad(M)$, where*

$$rad(M) = \max_{x \in M} \tau(x, \partial M).$$

Then $\mathcal{Z}_T$ determines (M, g, α) uniquely.

When $\alpha = 1$, a local version of Theorem 1.1 near ∂M is proven in [2]. The method to recover (M, g) from $\mathcal{Z}_T$, $T > 4\mathrm{diam}(M)$, for an arbitrary scalar α, is in [10].

Returning to $\mathbb{R}^3$, we observe that $\mathcal{Z}_T$ corresponds to the map

$$\mathcal{Z}_T : n \times E|_{\partial\mathcal{N} \times [0,T]} \to n \times H|_{\partial\mathcal{N}\times[0,T]},$$

which is, indeed, a well-known admittance map. Consider two copies of $\mathcal{N}$ as Riemannian manifolds with the metrics g and $\widetilde{g}$ of form (5), where, in case of $\widetilde{g}$, we use $\widetilde{\varepsilon}^j_k$, $\widetilde{\mu}^j_k$, $\widetilde{\alpha}$, and distance functions $\tau(x, y)$, $\widetilde{\tau}(x, y)$.

Theorem 1.2 *The group of transformations for the Maxwell system (1) – (3) with scalar wave impedance, which preserve the admittance map,*

$$\mathcal{Z}_T, \quad T > 2 \max_{x \in \mathcal{N}} \max(\tau(x, \partial\mathcal{N}), \widetilde{\tau}(x, \partial\mathcal{N})),$$

is generated by the group of diffeomorphisms, $X : \mathcal{N} \to \mathcal{N}$, $X|_{\partial\mathcal{N}} = id|_{\partial\mathcal{N}}$. The transformation formulae for ε, μ are then

$$\widetilde{\varepsilon}^{ij}(\widetilde{x}) = \frac{1}{det(DX)} \frac{\partial \widetilde{x}^i}{\partial x^k} \frac{\partial \widetilde{x}^j}{\partial x^l} \varepsilon^{kl}(x), \quad \widetilde{\mu}^{ij}(\widetilde{x}) = \frac{1}{det(DX)} \frac{\partial \widetilde{x}^i}{\partial x^k} \frac{\partial \widetilde{x}^j}{\partial x^l} \mu^{kl}(x), \tag{19}$$

where $\widetilde{x} = X(x)$ **and** *$\varepsilon^{ij} = \varepsilon^i_k g_0^{jk}$*.

The form (19) of admissible transformation for the two-dimensional conductivity problem is observed in [19] with relations between the low-frequency limit of the admittance map and the conductivity problem analysed in [12].

In this paper, we give a brief sketch of the proof of Theorems 1.1, emphasizing the part on α, and 1.2 (see [10] for more details.)

2 Reconstruction of the manifold and the metric

In this section we describe very briefly the method to determine (M, g) from $\mathcal{Z}_T$ referring to [10], [11] for further details. The basic analytical ideas, formulated in two theorems below, make possible to find the energy and location of an electromagnetic wave generated by a boundary source f.

Let $\omega(t) = (\omega^1(t), \omega^2(t))$ satisfies (7), (8), i.e. represents an electromagnetic wave in the absence of internal sources. *Electric and magnetic components* of the total energy, $\mathcal{E}(t)$, are given, respectively, as

$$\mathcal{E}_e(t) = \frac{1}{2}\int_M \frac{1}{\alpha}\,\omega^1(t) \wedge *\omega^1(t) = \frac{1}{2}\|\omega^1(t)\|_{L^2(\Omega^1 M)}; \tag{20}$$

$$\mathcal{E}_m(t) = \frac{1}{2}\int_M \frac{1}{\alpha}\,\omega^2(t) \wedge *\omega^2(t) = \frac{1}{2}\|\omega^2(t)\|_{L^2(\Omega^2 M)}, \tag{21}$$

$$\mathcal{E}(t) = \mathcal{E}_e(t) + \mathcal{E}_m(t).$$

with the rhs of (20), (21) defining the norms in L^2-spaces of $1-$ and $2-$ forms on M.

Theorem 2.1 *Let $\omega^f(t)$ be a solution of the Maxwell equations (7), (8), (11), (12) with a smooth boundary source f. Then*

1. *For any $T > 0$, $\mathcal{Z}_{2T}$ determines $\mathcal{E}^f_e(t), \mathcal{E}^f_m(t)$, $t \le T$.*
2. *For any $T > 0$, $\mathcal{Z}_T$ determines $\mathcal{E}^f(t)$, $t \le T$.*

It is also clear from this theorem that $\mathcal{Z}_{2T}$ determines the inner products $((\omega^f)^1(t), (\omega^g)^1(s))$ and $((\omega^f)^2(t), (\omega^g)^2(s))$, $s, t \le T$, where $\omega^g(s)$ is the wave generated by a boundary source g.

Proof: Maxwell system (7), (8), (12) implies that

$$\partial_t \mathcal{E}(\omega^f)(t) = \int_{\partial M} \mathbf{n}(\omega^f)^2(t) \wedge (\omega^f)^1(t) = \int_{\partial M} (\mathcal{Z}_T f)(t) \wedge f(t).$$

As $\mathcal{E}(0) = 0$, this proves part *2.*. For part *1.*, we refer to [10], [11]. □

To formulate the second result, we need some auxiliary notions. Let $\Gamma \subset \partial M$ be open. *The domain of influence* of Γ at time τ, $M(\Gamma, \tau)$ is given by

$$M(\Gamma, \tau) = \{x \in M \,|\, \tau(x, \Gamma) < \tau\}, \tag{22}$$

and *the double cone of influence*, $K(\Gamma, \tau)$, by

$$K(\Gamma, \tau) = \{(x, t) \in M \times [0, 2\tau] \,|\, \tau(x, \Gamma) < \tau - |\tau - t|\}. \tag{23}$$

Let also

$$X(\Gamma,\tau) = \mathrm{cl}_{L^2}\{(\omega^f)^1(\tau) \mid f \in \overset{\circ}{C}{}^\infty(]0,\tau[,\Omega^1\Gamma)\} \tag{24}$$

and

$$Y(\tau) = \mathrm{cl}_{L^2}\{((\omega_t^f)^1(\tau),\ (\omega_t^f)^2(\tau)) \mid f \in C_0^\infty(]0,\tau[,\Omega^1\partial M)\}, \tag{25}$$

where $\Omega^1\Gamma \subset \Omega^1\partial M$ consists of 1−forms with support in Γ.

Theorem 2.2 *1. Let $\omega(t)$ be a solution of the Maxwell equations (7), (8) such that*

$$\mathbf{t}\omega^1|_{\Gamma\times[0,2\tau]} = 0, \quad \mathbf{n}\omega^2|_{\Gamma\times[0,2\tau]} = 0.$$

Then $\partial_t\omega(t) = 0$ in $K(\Gamma,\tau)$.

2. *Let $X(\Gamma,\tau)$ be of form (24). Then,*

$$\delta_\alpha H_0^1(\Omega^2 M(\Gamma,T)) \subset X(\Gamma,T) \subset \mathrm{cl}_{L^2}\bigg(\delta_\alpha H(\delta_\alpha,\Omega^2 M(\Gamma,T))\bigg).$$

3. *Let $\tau > 2rad(M)$. Then*

$$Y(\tau) = \delta_\alpha H(\delta_\alpha,\Omega^2 M) \times d\overset{\circ}{H}(d,\Omega^1 M) = Y.$$

Here $H(\delta_\alpha,\Omega^2 M)$, $H(d,\Omega^1 M)$ are natural domains of operators δ_α and d in $L^2(\Omega^2 M)$ and $L^2(\Omega^1 M)$, correspondingly, and $\overset{\circ}{H}(d,\Omega^1 M) \subset H(d,\Omega^1 M)$ is defined by $\mathbf{t}\omega^1 = 0$. $H^s(\Omega^i M)$, $s \in \mathbb{Z}_+$, $i = 1,2$ is a Sobolev space of 1− and 2−differential forms, with $H_0^s(\Omega^i M) = \mathrm{cl}_{H^s}(\Omega^i M^{\mathrm{int}})$, where $\Omega^i M^{\mathrm{int}}$ consists of i−forms vanishing near ∂M. Furthermore, solution $\omega(t)$ in *1.* may be a weak solution of the Maxwell system (see e.g. [10], [11].)

The subspace $Y = \mathrm{Ran}(\mathcal{M}_e)$, where the operator $\mathcal{M}_e$ is defined by (15) on $\overset{\circ}{H}(d,\Omega^1 M) \times H(\delta_\alpha,\Omega^2 M)$. Operator $\mathcal{M}_e$ is not elliptic but the operator (15) with Dirichlet boundary condition, $\mathbf{t}\omega = 0$, considered as operator on $L^2(\mathbf{\Omega} M)$ is elliptic. Taking into account that, on Y, $\mathcal{M}_e$ and the operator (15) coincide, it is possible to use elliptic theory to study $\mathcal{M}_e|_Y$, [10].

It is standard in PDE-control to introduce the spaces of *generalised boundary sources*, $\mathcal{F}([0,T])$. To this end, we start with the equivalence,

$$f \sim g \quad \text{iff } \omega^f(t) = \omega^g(t), \text{ for } t > T,$$

which gives rise to the factor-space, $C_0^\infty([0,T],\Omega^1\partial M)/\sim$, and then complete it in the norm

$$|||f|||^2 = ||\partial_t(\omega^f)^1(t)||^2 + ||\partial_t(\omega^f)^2(t)||^2. \tag{26}$$

Clearly, rhs in (26) is independent of $t > T$. When $T > 2\text{rad}(M)$, it follows from Theorem 2.2, *3.*, that the delay operator, $f(\cdot) \to f(\cdot - \sigma)$ is well-defined on $\mathcal{F}([0,T])$ for small $\sigma > 0$. Therefore, we can define, in a natural way, the operator of $t-$ differentiation, $\mathbb{D}$ in this space. The domains of powers $\mathbb{D}^s$, $s \in \mathbb{Z}_+$ of $\mathbb{D}$, which we denote by $\mathcal{F}^s([0,T])$, may be characterized by

$$f \in \mathcal{F}^s([0,T]) \quad \text{iff} \quad \partial_t \omega^f \in \bigcap_{j=0}^{s} C^{s-j}(]T,\infty[, D(\mathcal{M}_e^j) \cap Y). \tag{27}$$

There is a natural identification between $\mathcal{F}^s([0,T_1])$ and $\mathcal{F}^s([0,T_2])$, $T_1, T_2 > 2\text{rad}(M)$,

$$f_1 \sim f_2 \quad \text{iff } \omega^{f_1}(t) = \omega^{f_2}(t) \text{ for } t > \max(T_1, T_2),$$

where $f_i \in \mathcal{F}([0,T_i])$, $i = 1,2$. Later, we will often write just $\mathcal{F}^s$ when reference to the time interval $]0,T[$ is irrelevant.

Theorems 2.1, 2.2 make possible to continue $\mathcal{Z}_T$ to $t > T$. The construction below is a straightforward extension of the one for the scalar wave equation [7], [9]. Another continuation method is described in [3].

Corollary 2.3 *$\mathcal{Z}_T$, $T > 2rad(M)$, uniquely determines $\mathcal{Z}_t$ for any $t > 0$.*

Proof: Let $2\varepsilon = T - 2\text{rad}(M)$. Due to Theorem 2.2, *3.*, for $f \in C_0^\infty([0,T];\, \Omega^1 \partial M)$, there is a sequence $f_n \in C_0^\infty([\varepsilon,T];\, \Omega^1 \partial M)$ with

$$\partial_t \omega^{f_n}(T) \to \partial_t \omega^f(T) \quad \text{in } L^2(\Omega^1 M) \times L^2(\Omega^2 M), \tag{28}$$

which is equivalent to the equation

$$\mathcal{E}(\omega^{g_n})(T) \to 0, \quad g_n = \partial_t (f - f_n). \tag{29}$$

In turn, due to Theorem 2.1, *2.*, equation (29) can be verified using $\mathcal{Z}_T$.

As $f, f_n = 0$ for $t \geq T$, it follows from (28) that

$$\mathbf{n}\partial_t (\omega^{f_n})^2|_{\partial M \times]T,\infty[} \to \mathbf{n}\partial_t (\omega^f)^2|_{\partial M \times]T,\infty[}. \tag{30}$$

As $\mathbf{n}\partial_t(\omega^{f_n})^2|_{\partial M \times]T,T+\varepsilon[}$ are known from $\mathcal{Z}_T$, (30) defines $\mathbf{n}(\omega^f)^2|_{\partial M \times]T,T+\varepsilon[}$. Iterating this procedure, we construct $\mathcal{Z}_\tau$ for any $\tau > 0$.

□

In further constructions, we will need $\mathcal{Z}_T$ with various $T > 2\text{rad}M$. Taking into account corollary 2.3, we will just speak about the admittance map $\mathcal{Z}$.

Let now $\Gamma_j \subset \partial M$ be open disjoint sets, $1 \leq j \leq J$ and τ_j^- and τ_j^+ be positive times with

$$0 < \tau_j^- < \tau_j^+ \leq \text{diam}(M), \quad 1 \leq j \leq J.$$

We define the set $S = S(\{\Gamma_j, \tau_j^-, \tau_j^+\}) \subset M$ given as an intersection of slices,

$$S = \bigcap_{j=1}^{J} \left(M(\Gamma_j, \tau_j^+) \setminus M(\Gamma_j, \tau_j^-) \right). \tag{31}$$

A rather technical construction, [10], makes possible to check whether $\mathrm{meas}(S) > 0$. To this end, let

$$\mathcal{F}_S(T_1) \subset \mathcal{F}^\infty = \bigcap_{s \geq 0} \mathcal{F}^s, \quad T_1 > 2\mathrm{rad}(M) + \mathrm{diam}(M),$$

be a subspace of generalised sources, f, such that

$$(\partial_t \omega^f)^1(T_1) \in X(\Gamma_j, \tau_j^+), \quad (\partial_t \omega^f)^2(T_1) = 0, \quad \partial_{tt}\omega^f(T_1) = 0 \text{ in } X(\Gamma_j, \tau_j^-). \tag{32}$$

The following theorem, based on theorems 2.1 and 2.2, is crucial for our considerations, see [10].

Theorem 2.4 *Let S and $\mathcal{F}_S(T_1)$ are defined as above. Then,*

1. *$\mathrm{meas}(S) = 0$ iff $\mathcal{F}_S(T_1) = \{0\}$;*
2. *Given $\mathcal{Z}$, it is possible to verify if $f \in \mathcal{F}_S(T_1)$ or not.*

Further steps are the same as in the case of a scalar wave equation [7]. They consist of constructing, using various $S(\{\Gamma_j, \tau_j^-, \tau_j^+\}) \subset M$, of the set of *boundary distance functions*, $\mathcal{R}(M)$,

$$\mathcal{R}(M) = \{r_x \in C(\partial M) \mid x \in M\}, \quad r_x : \partial M \to \mathbb{R}_+, \; r_x(z) = \tau(x, z),$$

and defining a Riemannian structure on $\mathcal{R}(M)$ which makes it isometric to (M, g).

3 Focusing sources

Our next goal is to find sequences of generalised sources, $\{f_p\}$, such that the corresponding waves, $\{\omega_p(t)\}$, at time $t = T_1$, converge to a $\delta-$type distribution concentrated at a point $y \in M^{\mathrm{int}}$. Let S_p of form (31) converge to y, i.e.

$$S_{p+1} \subset S_p, \quad \bigcap_{p>0} S_p = \{y\},$$

with

$$J(p) = 3, \quad \Gamma_j^{p+1} \subset \Gamma_j^p, \; \bigcap \Gamma_j^p = \{z_j\} \in \partial M, \quad \tau_j^{-,p}, \tau_j^{+,p} \to \tau(y, z_j) \tag{33}$$

(see [7] for the existence of such sequence). For a given sequence of $f_p \in \mathcal{F}_{S_p}(T_1)$, we can verify via $\mathcal{Z}$ if the corresponding waves,

$$\partial_t \omega_p(T_1) \to A_y, \tag{34}$$

where A_y is a distribution-form concentrated in y. Indeed, Theorem 2.1, *1.* makes possible to verify the existence of the limit,

$$\lim_{p\to\infty} \left(\partial_t \omega_p(T_1),\, \partial_t \omega^g(T_1)\right) \quad \text{for any } g \in \mathcal{F}^\infty. \tag{35}$$

Due to Theorem 2.2, *3.*, this is equivalent to the existence of $\lim_{p\to\infty} \left(\partial_t \omega_p(T_1),\, \eta\right)$ for any $\eta \in \bigcap_{s\geq 0} D(\mathcal{M}_e^s) \cap Y$. As, on the other hand, $\left(\partial_t \omega_p(T_1),\, \widetilde{\eta}\right) = 0$ for $\widetilde{\eta} \perp Y$, it is enough to verify (34) on $\partial_t \omega^g(T_1)$, $g \in \mathcal{F}^\infty$. In the future, we refer to the described sequences $\{f_p\}$ as *focusing sequences.*

Further information about A_y is given in the following Theorem.

Theorem 3.1 *Let $\{f_p\}$ be a focusing sequence, i.e. $\partial_t \omega_p(T_1) \to A_y$. Assume, in addition, that, for any $g \in \mathcal{F}^3$, there exists the limit (35). Then,*

$$A_y = (\delta_\alpha(\lambda \underline{\delta}_y), 0), \tag{36}$$

where $\lambda \in \Lambda^2 T_y^* M$ and $\underline{\delta}_y$ is the delta-function at y,

$$\int_M \frac{1}{\alpha} \omega^0 \wedge * \underline{\delta}_y = \left(\omega^0, \underline{\delta}_y\right) = \omega^0(y), \quad \omega^0 \in \Omega^0 M.$$

Proof: [1],

It follows from (32), (34) that

$$A_y = (A_y^1, 0), \quad \delta_\alpha A_y^1 = 0, \ \operatorname{supp}(dA_y^1) = \{y\}, \tag{37}$$

so that

$$\operatorname{supp}(\Delta_\alpha A_y^1) = \{y\}, \quad \Delta_\alpha = d\delta_\alpha + \delta_\alpha d. \tag{38}$$

As, due to (32),

$$\partial_t \omega(T_1) = 0 \quad \text{in} \quad M \setminus \left(\bigcap_{j=1}^{3} M(\Gamma_j^p, \tau_j^{+,p}) \right),$$

(33) implies that

$$A_y^1 = 0 \quad \text{in } M \setminus \left(\bigcap M(z_j, \tau(y, z_j)) \right).$$

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[1] Proof of this Theorem in [10] is incomplete.

By unique continuation for elliptic systems, this yields, together with (38), that

$$\mathrm{supp}(A^1_y) \subset \{y\}.$$

Furthermore, by our assumptions,

$$A_y \in (D(\mathcal{M}^s_e))' \subset \mathbf{H}^{-3}(M),$$

so that A^1_y may contain only $\underline{\delta}_y$ and its derivatives of the 1−st order,

$$A^1_y = \sum_{i,j=1}^{3} c^j_i \partial_j \underline{\delta}_y \, dx^i + \sum_{i}^{3} \widetilde{c}_i \underline{\delta}_y \, dx^i. \tag{39}$$

Substituting (39) into identity $\delta_\alpha A^1_y = 0$, we obtain (36). □

We note that, for any $\lambda \in \Lambda^2 T^*_y M$, there is a focusing sequence, $\{f_p\}$, such that the corresponding waves $\partial_t \omega_p(T_1)$ converge to $(\delta_\alpha(\lambda \underline{\delta}_y), 0)$. Indeed, let ϕ_p be a usual $\underline{\delta}_y$-sequence. Then, by Theorem 2.2, *3.*, there are $f_p \in \mathcal{F}^\infty$ with $\partial_t \omega^{f_p}(T_1) = (\delta_\alpha(\lambda \phi_p), 0)$. Then $\{f_p\}$ is a desired focusing sequence.

Let vary y, i.e. consider a family $\{f^y_p\}$, $y \in M^{\mathrm{int}}$, of focusing sources with

$$A_y = (\delta_\alpha(\lambda_y \underline{\delta}_y), 0), \quad \lambda_y \in \Lambda^2 T^*_y M. \tag{40}$$

Lemma 3.2 *Given $\mathcal{Z}$, it is possible to verify if the map $y \to \lambda_y$ determines a nowhere vanishing differential 2−form $\eta \in \Omega^2 M$,*

$$\eta(y) = \lambda_y.$$

Proof: Let $\phi \in \Omega^2 M$ has compact support. By Theorem 2.2, *3.*, there is $h \in \mathcal{F}^\infty$ with $\partial_t \omega^h(T_1) = (\delta_\alpha \phi, 0)$ By (34), (36),

$$\lim_{p \to \infty} \big(\partial_t \omega^y_p(T_1), \, \partial_t \omega^h(T_1)\big) = \big(\delta_\alpha(\lambda_y \underline{\delta}_y), \delta_\alpha \phi\big) = \langle \lambda_y, \, d\delta_\alpha \phi \rangle_y, \tag{41}$$

where $\omega^y_p(t)$ is the wave generated by f^y_p and $\langle \cdot, \cdot \rangle_y$ stands for the inner product in $\Lambda^2 T^*_y M$.

As ϕ is arbitrary, in a vicinity of any $y \in M^{\mathrm{int}}$ we can choose ϕ^y_i, $i = 1, 2, 3$, so that the 2−forms $d\delta_\alpha \phi^y_i$ are linearly independent near y. Indeed, if $(x^1, x^2, x^3), x(y) = 0$, are normal coordinates,

$$\phi_{ik} = \chi(x)(x^i)^2 \, dx^i \wedge dx^k, \; i < k,$$

where $\chi(x)$ is a cut-off function, satisfy desired conditions. Therefore, $\eta \in \Omega^2 M$ iff, for any $h \in \mathcal{F}^\infty$, the rhs in (41) defines a $C^\infty-$ function. □

It follows from this proof that, using 3 families, $\{f^{y,i}_p\}$, $i = 1, 2, 3$ of focusing sequences, we can verify via $\mathcal{Z}$, if the corresponding 2−forms η_i form a basis in

$\Lambda^2 T_y^* M$ for any $y \in M^{\mathrm{int}}$. Choosing $\xi_i \in \Omega^2(M)$, $i = 1, 2, 3$, which are linearly independent at any y, consider

$$K(y)(\partial_t \omega^f(t))^2 = \sum_{i=1}^{3} \langle \lambda_i(y),\, (\partial_t \omega^f(t))^2 \rangle_y\, \xi_i(y) \in \Omega^2(M^{\mathrm{int}}), \quad t \geq T_1. \tag{42}$$

Lemma 3.3 *Given $\mathcal{Z}$, it is possible to evaluate, for $f \in \mathcal{F}^\infty$, the $2-$form $K(y)(\partial_t \omega^f(t))^2$, where $K(y)$ is a smooth section of End$(\Omega^2 M)$.*

We note that, at this stage, $K(y)$ is unknown. Clearly, being able to identify families $\{f_p^{y,i}\}$, $i = 1, 2, 3$ with $K(y) = \mathrm{id}_y$, where id_y is the identity in $\Lambda^2 T_y^* M$, makes possible to find α. This will be done in the next section.

4 Green's form and reconstruction of α

Let again $\{h_p\}$ be a focusing sequence,

$$\lim_{p \to \infty} \partial_t \omega^{h_p}(T_1) = (\delta_\alpha(\mu_y \underline{\delta}_y), 0), \quad \mu \in \Lambda^2 T_y^* M. \tag{43}$$

Then,

$$\lim_{p \to \infty} \partial_t \omega^{h_p}(t + T_1) = G_m(x, t; y) = G_{m,\mu}(x, t; y), \tag{44}$$

where $G_{m,\mu}(x, t; y)$, called the *magnetic Green's function*, solves the problem

$$(\partial_t + \mathcal{M}) G_m(x, t; y) = 0, \ \mathbf{t} G_m(x, t; y) = 0, \ G_m(x, t; y)|_{t=0} = (\delta_\alpha(\mu_y \underline{\delta}_y), 0). \tag{45}$$

Using the WKB-method, we show the following lemma.

Lemma 4.1 *For $0 < t < \tau(y, \partial M)$,*

$$G_{m,\mu}(x, t; y) = \left((G_{m,\mu}(x, t; y))^1,\, (G_{m,\mu}(x, t; y))^2 \right),$$

where

$$(G_{m,\mu}(x, t; y))^1 = [*(*Q(x, y)\mu_y \wedge d_x \tau)]\, \underline{\delta}^{(2)}(t - \tau(x, y)) + r^1(x, t; y), \tag{46}$$

$$(G_{m,\mu}(x, t; y))^2 = [*(*Q(x, y)\mu_y \wedge d_x \tau) \wedge d_x \tau]\, \underline{\delta}^{(2)}(t - \tau(x, y)) + r^2(x, t; y). \tag{47}$$

Here $Q(x, y) \in End(\Lambda^2 T_y^* M^{int},\, \Lambda^2 T_x^* M^{int})$ is smooth outside $diag(M^{int})$ and $\underline{\delta}^{(2)}(t - \tau)$ is the second derivative of the δ-function on the sphere $S_y(t)$. Singularities of r^1, r^2 contain only $\underline{\delta}^{(1)}(t - \tau)$ and $\underline{\delta}(t - \tau)$.

Using (41) and Lemma 3.3, we see that $\mathcal{Z}$ determines

$$K(x)\ [*(*Q(x,y)\mu_y \wedge d_x\tau) \wedge d_x\tau]\,,$$

for any μ_y of form (43). If it happens that $K(x) = \mathrm{id}_x$, then

$$K(x)\,[*(*Q(x,y)\mu_y \wedge d_x\tau) \wedge d_x\tau]\,(v,w) = 0, \quad \text{for any } v,w \in T_xS_y(t). \tag{48}$$

Using this, we can impose (48) as a condition for the focusing sources $\{f_p^{y,i}\}$, that define endomorphism $K(x)$. Then we have

$$K(x) = c(x)\mathrm{id}_x,$$

for a nowhere vanishing $c \in C^\infty(M^{\mathrm{int}})$. Furthermore, if $K(x) = \mathrm{id}_x$, then

$$d\left[K(x)\,\left(\partial_t\omega^h\right)^2(x,t)\right] = 0, \quad \text{for any } h \in \mathcal{F}^\infty. \tag{49}$$

Imposing conditions (49) on $\{f_p^{y,i}\}$, we obtain that

$$c(x) = \mathrm{const}.$$

Returning to (42), we see that $\mathcal{Z}$ determine, for $h \in \mathcal{F}^\infty$,

$$c^2 \int_M (\partial_t\omega^h)^2(x,t) \wedge *(\partial_t\omega^h)^2(x,t).$$

Dividing it by $\mathcal{E}_m(\partial_t\omega^h)^2)(T_1)$ evaluated by Theorem 2.1, *1.*, and taking a sequence $\{h_p\}$ with $\mathrm{supp}(\partial_t\omega_p(T_1)) \to \{y\}$, we find $2c^2\alpha(y)$.

As $\mathcal{Z}(c\alpha) = c^{-1}\mathcal{Z}(\alpha)$, where $\mathcal{Z}(c\alpha)$ is the impedance map corresponding to $(M,g,c\alpha)$, we find α. This completes the proof of Theorem 1.1.

5 Proof of Theorem 1.2

Let $\varepsilon_k^j(x)$, $\mu_k^j(x)$ and $\widetilde{\varepsilon}_k^j(x)$, $\widetilde{\mu}_k^j(x)$, $x \in \mathcal{N} \subset \mathbb{R}^3$. It then follows from assumptions of Theorem 1.2, that, due to Theorem 1.1,

$$(\mathcal{N}, g, \alpha) \approx (\widetilde{\mathcal{N}}, \widetilde{g}, \widetilde{\alpha}),$$

where $\approx$ stands for isometry. Therefore, there is a diffeomorphism X,

$$X : \mathcal{N} \to \mathcal{N}, \quad X|_{\partial\mathcal{N}} = \mathrm{id}|_{\partial\mathcal{N}},$$

such that

$$\widetilde{g} = X_* g, \quad \widetilde{\alpha} = X_*\alpha. \tag{50}$$

As, due to (5),

$$\varepsilon_k^j = \frac{1}{\alpha}\sqrt{g}g^{jn}\,\delta_{nk}, \quad \mu_k^j = \alpha\sqrt{g}g^{jn}\,\delta_{nk},$$

(50) yields (19).

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