JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 137 (2001) 213–227

www.elsevier.com/locate/cam

Numerical solution of the Minkowski problem

L. Lamberg[a], M. Kaasalainen[b, *]

[a]Department of Mathematics, P.O. Box 4, Yliopistokatu 5, FIN-00014 University of Helsinki, Finland

[b]Observatory, P.O. Box 14, Tähtitorninmäki, FIN-00014 University of Helsinki, Finland

Received 1 December 1999; received in revised form 29 November 2000

Abstract

We present a numerical procedure for solving the Minkowski problem, i.e., determining the convex set corresponding to a given curvature function. The method is based on Minkowski's isoperimetric inequality concerning convex and compact sets in $\mathbb{R}^3$. The support function of the target set is approximated in finite function space, so the problem becomes one of constrained optimization in $\mathbb{R}^n$, which in turn is solved by Newtonian (or other) iteration. We prove some properties of the optimization function and the constraining set and present some numerical examples. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 52-04; 52A15

Keywords: Support function; Mixed volume; Curvature function; Spherical harmonics; Newton's method

1. Introduction

The problem of reconstructing a convex surface when its curvature is known, i.e., the Minkowski problem, has been known in mathematical literature since Minkowski's original paper [7,10,11]. Minkowski offered quite constructive proofs; nevertheless, no implementable numerical algorithm for the solution was presented until in [8,9]. That study concerned the reconstruction of a polyhedron when the areas of its facets are known; however, the algorithm was neither general nor provably convergent (mainly due to the slightly complicated concept of the volume function of a polyhedron). Two versions of an algorithm fulfilling these conditions (for convex polyhedra) were first given in [5], where the convergence of iteration was proven rigorously for the gradient method; Newton's method was described as well.

In this paper, we study the Minkowski problem in finite function space: instead of using polyhedra, the problem is discretized by employing truncated spherical harmonics series. This is a very robust

* Corresponding author. Fax: +358-9-191-22952.

E-mail address: kaselain@astro.helsinki.fi (M. Kaasalainen).

PII: S0377-0427(01)00360-0

and easily implementable method if the target function is smooth. Convergence cannot be proven rigorously, mostly because the nonnegativity of the iterated function cannot be guaranteed; however, we give some proofs and numerical results that corroborate the practicality of the scheme. Our main choice is Newton's method; we also present the gradient method because of its simplicity. The principal motive for our developing several algorithms has been the need for efficient methods of surface reconstruction in photomorphography, i.e., deducing the shape of an object from its total brightness in various viewing and illumination geometries ([2–4]).

In Section 2 we present some background and definitions. The numerical schemes (Newton's method and gradient method) and conclusions are given in Section 3, while some relevant lemmas and theorems whose detailed exposition is not required in the main text are presented in Section 4.

2. Background

The sets $\mathscr{K}, \mathscr{R}, \mathscr{Q}, \ldots$ are always convex and compact sets in $\mathbb{R}^3$ with interior points, i.e. with nonzero volumes, unless stated otherwise. The volume of a set is the Lebesgue measure. Positivity of volume implies that the geometric centroid of a set is an interior point. The notation ω always denotes a unit vector of $\mathbb{R}^3$. *The support function* of a set $\mathscr{K}$ (not necessarily convex) is denoted by $\rho_{\mathscr{K}}$. Its characteristic properties can be found in [5]. The support function $\rho_{\mathscr{K}}: S^2 \to \mathbb{R}$ is defined as a function of directions $\omega \in S^2 = \{\mathbf{x} \in \mathbb{R}^3 \mid ||\mathbf{x}|| = 1\}$ by the formula

$$\rho_{\mathscr{K}}(\omega) = \sup\{x^T\omega \mid x \in \mathscr{K}\}, \quad \omega \in S^2. \tag{2.1}$$

In fact, it follows from compactness that the supremum in (2.1) is a maximum, so there exists at least one point $x \in \mathscr{K}$, for which $\rho_{\mathscr{K}}(\omega) = x^T\omega$. It is easy to see that $\rho_{\mathscr{K}}$ is a continuous function on S^2 [5]. The support function $\rho_{\mathscr{K}}$ can also be defined as a positively homogeneous function $\rho_{\mathscr{K}}: \mathbb{R}^3 \to \mathbb{R}$ of degree one by the formula

$$\rho_{\mathscr{K}}(y) = \sup\{y^Tx \mid x \in \mathscr{K}\}, \quad y \in \mathbb{R}^3.$$

We shall use the definition (2.1) in this paper. The support function $\rho_{\mathscr{K}}$ fully determines the convex and compact set $\mathscr{K}$:

$$\mathscr{K} = \bigcap_{\omega \in S^2} \{x \in \mathbb{R}^3 \mid x^T\omega \leqslant \rho_{\mathscr{K}}(\omega)\}. \tag{2.2}$$

It satisfies two characteristic conditions given by Minkowski: if any function ρ satisfies the conditions, the formula (2.2) defines a nonempty convex and compact set $\mathscr{R}$ and we have $\rho_{\mathscr{R}} = \rho$. If the set $\mathscr{K}$ is not convex, the intersection (2.2) gives the convex hull of $\mathscr{K}$. When the set $\mathscr{K}$ is translated by a vector $r \in \mathbb{R}^3$, we get the set $\mathscr{K} + r$ and

$$\rho_{\mathscr{K}+r}(\omega) = \rho_{\mathscr{K}}(\omega) + r^T\omega. \tag{2.3}$$

A positively linear sum $\rho = t_1\rho_1 + \cdots + t_n\rho_n$, $t_k \geqslant 0$, of support functions $\rho_1, \ldots, \rho_n$, which correspond to the sets $\mathscr{K}_1, \ldots, \mathscr{K}_n$, is also a support function so, by the formula (2.2), it defines a convex and compact set $\mathscr{K}$. It is easy to see that $\mathscr{K} = t_1\mathscr{K}_1 + \cdots + t_n\mathscr{K}_n$ in the usual sense, as a sum of sets in the vector space $\mathbb{R}^3$.

The quantity *mixed volume* $V(\mathscr{K},\mathscr{R},\mathscr{Q})$, introduced by Minkowski, is associated with the three sets $\mathscr{K},\mathscr{R}$ and $\mathscr{Q}$ (definition in [6]). It has the characteristic property: if $\mathscr{W}=t_1\mathscr{K}+t_2\mathscr{R}+t_3\mathscr{Q},\ t_k\geqslant 0$, we have a homogeneous cubic polynomial which gives the volume of the set $\mathscr{W}$:

$$\begin{aligned}V(\mathscr{W}) &= t_1^3V(\mathscr{K},\mathscr{K},\mathscr{K})+3t_1^2t_2V(\mathscr{K},\mathscr{K},\mathscr{R})+3t_1t_2^2V(\mathscr{K},\mathscr{R},\mathscr{R})\\ &\quad +t_2^3V(\mathscr{R},\mathscr{R},\mathscr{R})+6t_1t_2t_3V(\mathscr{K},\mathscr{R},\mathscr{Q})+\cdots+t_3^3V(\mathscr{Q},\mathscr{Q},\mathscr{Q}).\end{aligned} \tag{2.4}$$

Thus mixed volumes are the coefficients of this polynomial. Especially $V(\mathscr{K},\mathscr{K},\mathscr{K})$ is the volume of the set $\mathscr{K}$ and $V(B^3,\mathscr{K},\mathscr{K})=1/3\,A(\mathscr{K})$, a third of the surface area of $\mathscr{K}$ (B^3 is the unit ball in $\mathbb{R}^3$). The mixed volume brings volumes and areas under the same concept. $V(\mathscr{K},\mathscr{R},\mathscr{Q})$ is invariant under permutations and translations of the sets $\mathscr{K},\mathscr{R}$ and $\mathscr{Q}$. In the space of convex and compact sets, the metric d is defined as usual by (equivalent definitions on page 148 in [6])

$$d(\mathscr{K},\mathscr{R})=\sup_{\omega\in S^2}|\rho_{\mathscr{K}}(\omega)-\rho_{\mathscr{R}}(\omega)|=||\rho_{\mathscr{K}}-\rho_{\mathscr{R}}||_\infty .$$

The mixed volume is continuous in the sense that

$$\lim_{k\to\infty}V(\mathscr{K}_k,\mathscr{R}_k,\mathscr{Q}_k)=V(\mathscr{K},\mathscr{R},\mathscr{Q}),$$
$$\text{if}\ \lim_{k\to\infty}\mathscr{K}_k=\mathscr{K},\quad \lim_{k\to\infty}\mathscr{R}_k=\mathscr{R}\quad\text{and}\quad \lim_{k\to\infty}\mathscr{Q}_k=\mathscr{Q}.$$

If $t\geqslant 0$, then $V(t\mathscr{K},\mathscr{R},\mathscr{Q})=tV(\mathscr{K},\mathscr{R},\mathscr{Q})$. Let the volumes of the sets $\mathscr{K},\mathscr{R}$ and $\mathscr{Q}$ be $V_{\mathscr{K}},V_{\mathscr{R}}$ and $V_{\mathscr{Q}}$. Note that $V(V_{\mathscr{K}}^{-1/3}\mathscr{K},V_{\mathscr{K}}^{-1/3}\mathscr{K},V_{\mathscr{K}}^{-1/3}\mathscr{K})=1$. Minkowski has proved the following fundamental theorem about the mixed volume [10]:

Theorem A. $V(V_{\mathscr{K}}^{-1/3}\mathscr{K},V_{\mathscr{R}}^{-1/3}\mathscr{R},V_{\mathscr{Q}}^{-1/3}\mathscr{Q})\geqslant 1$, *and the equality holds if and only if the sets* $\mathscr{K},\mathscr{R}$ *and* $\mathscr{Q}$ *are homothetic.*

Minkowski has also proved a sharper adaptation of the mixed volume theorem concerning $V(\mathscr{K},\mathscr{R},\mathscr{R})$, which we will use in what follows. Let us assume that the centroids of the sets $\mathscr{K}$ and $\mathscr{R}$ are at the origin. The geometric centroid x_0 of the set $\mathscr{K}$ is defined by the equation $\int_{\mathscr{K}}(x-x_0)\,\mathrm{d}x=0$. Let ρ_1 and ρ_2 be the support functions of sets $V_{\mathscr{K}}^{-1/3}\mathscr{K}$ and $V_{\mathscr{R}}^{-1/3}\mathscr{R}$: $\rho_1=V_{\mathscr{K}}^{-1/3}\rho_{\mathscr{K}}$ and $\rho_2=V_{\mathscr{R}}^{-1/3}\rho_{\mathscr{R}}$. Since the centroids are at the origin, support functions ρ_1 and ρ_2 are positive everywhere, and they have positive minima and maxima. We define

$$D=\sup_{\omega\in S^2}\frac{\rho_1(\omega)}{\rho_2(\omega)},\quad d=\inf_{\omega\in S^2}\frac{\rho_1(\omega)}{\rho_2(\omega)}.$$

Theorem B (Minkowski [10]).

$$V(V_{\mathscr{K}}^{-1/3}\mathscr{K},V_{\mathscr{R}}^{-1/3}\mathscr{R},V_{\mathscr{R}}^{-1/3}\mathscr{R})\geqslant 1+\kappa\frac{(D-1)^6}{D^5},$$

where the value of the Minkowski constant is $\kappa=2^{-10}\times 3^{-4}\times 7^{-4/3}$. *In addition* $D^2-1\geqslant\kappa(1-d)^6/d$.

Theorem B can be interpreted as follows: Let a set $\mathscr{R}$ be fixed and let $\mathscr{K}$ be chosen from the family of convex and compact sets with volumes $\geqslant 1$. The mixed volume functional $J(\mathscr{K}):=V(\mathscr{K},\mathscr{R},\mathscr{R})$

attains its minimum ($=V_{\mathscr{R}}^{2/3}$) exactly when $\mathscr{K}$ is homothetic with the set $\mathscr{R}$ and the volume is $V_{\mathscr{K}}=1$. We will use this fact in the numerical solution of the Minkowski problem, which can be interpreted by Theorem B as a constrained optimization problem. Theorem B will also give a numerical criterion for the goodness of fit of an approximation. On the other hand, the mixed volume functional can be represented as the integral [5]

$$V(\mathscr{K},\mathscr{R},\mathscr{R})=1/3\int_{S^2}\rho_{\mathscr{K}}(\omega)\mu_{\mathscr{R}}(\mathrm{d}\omega), \tag{2.5}$$

where $\mu_{\mathscr{R}}$ is *the curvature function* of the set $\mathscr{R}$: it is the uniquely determined Radon measure on S^2 with the property (2.5) for every convex and compact set $\mathscr{K}$ with interior points. In the case of a strictly convex $\mathscr{R}$ the (positive) Gaussian curvature of the boundary can be expressed as a function $K_{\mathscr{R}}:S^2\to\mathbb{R}_+$, where *the variable* $\omega\in S^2$ *is the outward normal of the boundary*. Then $\mu_{\mathscr{R}}=K_{\mathscr{R}}^{-1}$. Furthermore, if $\rho:S^2\to\mathbb{R}$ is a C^2-function, we have an equation of the Monge–Ampère type

$$\mu=A(\rho)C(\rho)-B(\rho)^2, \tag{2.6}$$

where using spherical coordinates (θ,ϕ) on S^2,

$$\begin{aligned}
A(\rho)&=\frac{\partial^2\rho}{\partial\theta^2}+\rho,\\
B(\rho)&=\frac{1}{\sin\theta}\frac{\partial^2\rho}{\partial\theta\partial\phi}-\frac{\cos\theta}{\sin^2\theta}\frac{\partial\rho}{\partial\phi},\\
C(\rho)&=\frac{1}{\sin^2\theta}\frac{\partial^2\rho}{\partial\phi^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial\rho}{\partial\theta}+\rho.
\end{aligned}$$

According to Minkowski ρ is a support function if and only if $\mu(\omega)\geqslant 0\ \forall\omega\in S^2$ in (2.6) (see also continuity properties [5]). Then $\mu=\mu_{\mathscr{R}}$, where ρ is the support function of $\mathscr{R}$. Eq. (2.6) has also been solved directly [11].

We list the characteristic properties of curvature functions [5]:

Proposition. *The curvature function* μ *has the following properties*:
(a) μ *is a positive regular measure*,
(b) $\int_{S^2}\omega\,\mathrm{d}\mu(\omega)=0$,
(c) *The support of the measure* μ *is not contained in any single great circle of the sphere* S^2.

Summarizing, the conditions (a)–(c) of the Proposition are sufficient and necessary conditions for a Radon measure μ on S^2 to be a curvature function, as the Theorem below shows. We define $\mathscr{C}(\mathbb{R}^3)$ to be the space of all convex and compact sets with interior points and $\mathscr{M}(S^2)$ to be the space consisting of all Radon measures on the sphere S^2 satisfying conditions (a)–(c) in the Proposition. The following theorem, which solves the Minkowski problem, can be found in Ref. [5], for example.

Theorem. *Let* $\mu\in\mathscr{M}(S^2)$. *Then there is* $\mathscr{R}\in\mathscr{C}(\mathbb{R}^3)$, *which is unique except for translations, so that* $\mu_{\mathscr{R}}=\mu$.

3. Numerical procedure

In this section we present a numerical procedure for computing $\mathscr{R} \in \mathscr{C}(\mathbb{R}^3)$ when its curvature function $\mu_{\mathscr{R}} \in \mathscr{M}(S^2)$ (briefly μ) is known. More precisely: we compute approximately the support function $\rho_{\mathscr{R}}$ of $\mathscr{R}$ and we approximate it in a finite dimensional function space. The support functions ρ are continuous and they are defined on the sphere S^2. On the other hand, L^2-functions on S^2 can be expanded as series of spherical harmonics Y_l^m. If ρ is smooth enough, such a series converges not only in the sense of the L^2-norm but uniformly as well. According to Nirenberg ρ_μ is smooth when μ is positive and smooth. Then we can expect that a good approximation—in the sense of the supnorm, associated with the usual topology among convex sets—can be found in a space that is spanned by a finite number of spherical harmonics. Of course the number of terms that we need in the series varies from case to case. Thus we model support functions ρ by finite series of real normalized spherical harmonics: $\rho_{\mathbf{x}} = \sum_{i=1}^n x_i Y_i$ [1]. We use just one index i in notations. No Y_i is of the form Y_1^m (has degree one), since those terms correspond to translations of sets in $\mathbb{R}^3$. *This way translations will be eliminated immediately*. We fix an index $n \in \mathbf{N}_+$ and our unknown (parameter) is $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. A basic difficulty of such a model is that $\rho_{\mathbf{x}}$ is not necessarily a support function for all values $\mathbf{x} \in \mathbb{R}^n$ and thus it does not define a convex set. In any case it defines an $\mathbb{R}^3$-surface. The (global) parametrization of this surface X is

$$X(\theta, \phi) = M^T \left(\rho, \frac{\partial \rho}{\partial \theta}, \frac{1}{\sin\theta} \frac{\partial \rho}{\partial \phi} \right), \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi \leqslant 2\pi, \tag{3.1}$$

where

$$M = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}.$$

At a point $X(\theta, \phi)$ the surface X has the normal $\omega = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ (at irregular points we may not be able to speak about the normal). We can say that the surface is given in normal coordinates. Furthermore, $\rho(\theta, \phi)$ is the distance between the origin and the tangential plane at $X(\theta, \phi)$. The function $\rho_{\mathbf{x}}$ is a support function exactly when the surface X is convex.

By inserting $\rho_{\mathbf{x}} = \sum_{i=1}^n x_i Y_i$ in the Monge–Ampère equation (2.6) we obtain the curvature function $\mu_{\mathbf{x}}$ of the surface X. Strictly speaking $\mu_{\mathbf{x}}$ is a curvature function in the convex case; generally it is the inverse of the Gaussian curvature given in normal coordinates. This function is a homogeneous quadratic polynomial in $\mathbf{x}$. In order to compare it with the given curvature function μ, we expand them both as series of real spherical harmonics Y_i: $\mu = \sum_{i=1}^\infty c_i Y_i$ and $\mu_{\mathbf{x}} = \sum_{i=1}^\infty c_i(\mathbf{x}) Y_i$, where the coefficients $c_i(\mathbf{x})$ are homogeneous polynomials of $\mathbf{x}$:

$$c_i(\mathbf{x}) = \frac{1}{2} \sum_{j,k=1}^{n} M(i,j,k) x_j x_k \tag{3.2}$$

and the coefficients M are in turn integrals

$$M(i,j,k) = \int_{S^2} (A(Y_j)C(Y_k) - 2B(Y_j)B(Y_k) + A(Y_k)C(Y_j)) Y_i \, \mathrm{d}\omega. \tag{3.3}$$

They have the form of the mixed volumes presented by Minkowski. Thus, *they are invariant under permutations of indices i, j and k* (the proof is essentially the same as that of Minkowski [10]). Also, the expansions of $\mu_{\mathbf{x}}$ and μ have no $Y_i = Y_l^m$-terms with $l = 1$. We denote by $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$ the vector composed of the n first coefficients c_i. Correspondingly $\mathbf{c}(\mathbf{x}) = (c_1(\mathbf{x}), \ldots, c_n(\mathbf{x})) \in \mathbb{R}^n$. Let T be the plane in $\mathbb{R}^n$ defined by $\int_{S^2} \rho_{\mathbf{x}}\, \mathrm{d}\mu = \mathbf{x}^T\mathbf{c} = 1$, $T = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^T\mathbf{c} = 1\}$, and let $Z = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^T\mathbf{c} = 0\}$ be the corresponding linear subspace of $\mathbb{R}^n$. We denote by $K = \{\mathbf{x} \in T \mid \mu_{\mathbf{x}}(\omega) \geqslant 0\ \forall \omega \in S^2\}$ the set of "convex points" $\mathbf{x} \in T$, i.e., points defining convex sets. Let then $\mathscr{K}(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y}^T\omega \leqslant \rho_{\mathbf{x}}(\omega)\ \forall \omega \in S^2\}$ be the corresponding convex compact set. The set K is itself closed and convex and it has interior points. We define "the volume function" V on $\mathbb{R}^n$ by the formula

$$V(\mathbf{x}) = \frac{1}{3}\int_{S^2} \rho_{\mathbf{x}}\mu_{\mathbf{x}}\, \mathrm{d}\omega = \frac{1}{3}\sum_{i=1}^{n} x_i c_i(\mathbf{x}) = \frac{1}{6}\sum_{i,j,k=1}^{n} M(i,j,k) x_i x_j x_k, \quad \mathbf{x} \in \mathbb{R}^n. \tag{3.4}$$

It gives the correct volume of the set when $\mathbf{x}$ is a convex point; in other cases this may not hold true. We can consider our problem to be one of constrained optimization in which we want to maximize the function V in the set K. This maximization problem is consistent with finding a support function $\rho_{\mathbf{x}}$ that minimizes the inner product $\langle \rho_{\mathbf{x}}, \mu \rangle = \int_{S^2} \rho_{\mathbf{x}}\, \mathrm{d}\mu$ in the set $V(\mathbf{x}) \geqslant 1$. Thus, our approximation criterion follows directly from Minkowski's Theorem B. According to Theorem 1 (Section 4) the function V has a unique maximum point in the set K. We denote it by $\mathbf{x}^*$; it gives a support function which defines our n-approximation for the convex compact set corresponding to μ (except for size). Let us consider the convergence of this approximation when n is increased. For every $n \in \mathbf{N}_+$ let the maximum point of Theorem 1 be $\mathbf{x}_n^*$ in K_n. If μ is a C^∞-function and everywhere positive, Theorem 2 (Section 4) shows that $\mathscr{K}(\mathbf{x}_n^*) \xrightarrow{n} (3V_{\mathscr{R}})^{-1}\mathscr{R}$, where $\mu = \mu_{\mathscr{R}}$:

$$\lim_{n\to\infty} d(\mathscr{K}(\mathbf{x}_n^*), (3V_{\mathscr{R}})^{-1}\mathscr{R}) = \lim_{n\to\infty} \sup_{\omega \in S^2} |\rho_{\mathbf{x}_n^*}(\omega) - \rho(\omega)| = 0,$$

where $\rho = (3V_{\mathscr{R}})^{-1}\rho_{\mathscr{R}}$. Using Theorem B yields also an error estimate

$$\left(\frac{\mu_{\mathbf{x}_n^*}}{3\mu_{\mathscr{R}} V(\mathbf{x}_n^*)}\right)^{1/3} - 1 \approx \frac{1}{3} V_{\mathscr{R}}^{-2/3} V(\mathbf{x}_n^*)^{-1/3} - 1 \geqslant \kappa \frac{(D-1)^6}{D^5},$$

which unfortunately has no practical meaning because of the smallness of the Minkowski constant κ.

The constraints of the optimization problem of finding $\mathbf{x}^*$ (with a fixed n) can be interpreted as two conditions: the easy linear one that $\mathbf{x} \in T$, and the condition that the function $\mu_{\mathbf{x}}$ must be nonnegative everywhere. The latter peculiar one is theoretically difficult. Fortunately, it will often take care of itself in practice: Newton's iteration method (which we use) tends to converge to the nearest stationary point. Thus we begin the iteration with an interior point of K (for instance with $\mathbf{x}_0 = (1/c_1, 0, \ldots, 0)$). Theorem 3 (Section 4) shows that the function V (also $V^{1/3}$) has at most one T-constrained stationary point in K. When it exists, it is $\mathbf{x}^*$.

We will need a basis of the subspace Z. If for instance $c_n \neq 0$, then

$$\mathbf{x} \in Z \quad \Leftrightarrow \quad \mathbf{x}_n = -\sum_{k=1}^{n-1} (c_k/c_n) x_k \quad \Leftrightarrow \quad \mathbf{x} = \sum_{k=1}^{n-1} x_k(\mathbf{e}_k - (c_k/c_n)\mathbf{e}_n), \quad x_k \in \mathbb{R},$$

where $\mathbf{e}_k$, $k \in [n]$, are the usual basis vectors of $\mathbb{R}^n$. Obviously the sequence $(\mathbf{e}_k - (c_k/c_n)\mathbf{e}_n)_{k=1}^{n-1}$ is linearly independent and thus a basis of Z, so Gram–Schmidt orthonormalization produces an orthonormal basis $(\mathbf{a}_k)_{k=1}^{n-1}$, $\mathbf{a}_k=(a_{1k},\dots,a_{nk})$, of Z. Let $A=(a_{ik})$ be the corresponding $n\times(n-1)$ matrix.

Let $\mathbf{g}$ and H be, respectively, the gradient vector and the Hessian matrix of the function V, and let $\mathbf{g}_p$ and H_p be their projections onto subspace Z: $\mathbf{g}_p(\mathbf{x})=A^T\mathbf{g}(\mathbf{x})$ and $H_p(\mathbf{x})=A^TH(\mathbf{x})A$. Correspondingly, let $\tilde{\mathbf{g}}$, $\tilde{H}$ and $\tilde{\mathbf{g}}_p$, $\tilde{H}_p$ be these quantities with respect to the function $V^{1/3}$:

$$\tilde{\mathbf{g}}(\mathbf{x})=1/3\,V(\mathbf{x})^{-2/3}\mathbf{g}(\mathbf{x}),$$

$$\tilde{H}(\mathbf{x})=1/3\,V(\mathbf{x})^{-2/3}H(\mathbf{x})-2/9\,V(\mathbf{x})^{-5/3}\mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})^T,$$

$$\tilde{\mathbf{g}}_p(\mathbf{x})=A^T\tilde{\mathbf{g}}(\mathbf{x})=1/3\,V(\mathbf{x})^{-2/3}A^T\mathbf{g}(\mathbf{x}),$$

$$\tilde{H}_p(\mathbf{x})=A^T\tilde{H}(\mathbf{x})A=1/3\,V(\mathbf{x})^{-2/3}A^T[H(\mathbf{x})-2/3\,V(\mathbf{x})^{-1}\mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})^T]A.$$

They are defined in K since then $V(\mathbf{x})>0$ by Lemma 1 (Section 4). According to Lemma 4 (Section 4) we have $\mathbf{g}(\mathbf{x})=\mathbf{c}(\mathbf{x})$ and $H_{ij}(\mathbf{x})=\sum_{k=1}^n M(i,j,k)x_k$, $i,j\in[n]$.

In Newton's method, applied to a linearly constrained problem to maximize the function V in the plane T, a new iteration point is computed using formulas

$$\mathbf{x}_{k+1}=\mathbf{x}_k+t_kA\mathbf{d}_k,\quad H_p(\mathbf{x}_k)\mathbf{d}_k=-\mathbf{g}_p(\mathbf{x}_k), \tag{3.5}$$

where $A\mathbf{d}_k$, $\mathbf{d}_k\in\mathbb{R}^{n-1}$, is the direction vector and $t_k\in\mathbb{R}$ is the length parameter of the iteration step. The number n of unknowns is often large and therefore we have a good reason to solve the linear equation in (3.5) approximately using, e.g., the conjugate gradient method. A problem arises immediately: is the matrix H_p definite or nonsingular at all? Possibly not always, but we can modify Newton's method as follows: The problem of finding the maximum point of the volume function V is consistent with finding the maximum point of the function $V^{1/3}$. Theorem 4 (Section 4) says that $\tilde{H}_p(\mathbf{x})$ is negative definite in the interior of K. The Newtonian direction $A\mathbf{d}_k$ can be chosen, always or sometimes, by solving the linear equation $\tilde{H}_p(\mathbf{x}_k)\mathbf{d}_k=-\tilde{\mathbf{g}}_p(\mathbf{x}_k)$ instead of the equation in (3.5). Note that $\mathbf{d}_k$ is then a slope direction: $(\mathbf{d}_k)^T\tilde{\mathbf{g}}_p(\mathbf{x}_k)>0$. The equation can be written as

$$A^T(H(\mathbf{x}_k)-2/3\,V(\mathbf{x}_k)^{-1}\mathbf{g}(\mathbf{x}_k)\mathbf{g}(\mathbf{x}_k)^T)A\,\mathbf{d}_k=-A^T\mathbf{g}(\mathbf{x}_k). \tag{3.6}$$

Because $V(\mathbf{x}+tA\mathbf{d})=V(\mathbf{x})+\mathbf{d}^TA^T\mathbf{g}(\mathbf{x})\,t+1/2\,\mathbf{d}^TA^TH(\mathbf{x})A\mathbf{d}\,t^2+V(A\mathbf{d})t^3$, we choose the maximum point of this polynomial as the step length t_k (it has to be a positive real number).

Let us assume that $\mathbf{x}^*$ is an interior point of K. Then it is a stationary point of V ($V^{1/3}$). It is also a strong maximum of V ($V^{1/3}$), which follows immediately from Theorem 3: then $H_p(\mathbf{x}^*)$ and $\tilde{H}_p(\mathbf{x}^*)$ are negative definite. This fact contributes remarkably to the convergence rate. In order to obtain V, $\mathbf{g}$ and H, we have to compute many integrals $M(i,j,k)$ (a number $\sim 1/6\,n^3$, most of them zero). However, they are always the same so they can be computed in advance and stored.

Summarizing, the iteration procedure is as follows:

1. *Compute* $M(i,j,k),\quad i,j,k\in[n]$
2. *Construct* $\mathbf{c}=(c_1,\dots,c_n),\ c_i=\int_{S^2}Y_i(\omega)\,\mathrm{d}\mu(\omega),\quad i\in[n]$
3. *Compute the basis matrix* $A\in\mathbb{R}^{n\times(n-1)}$ *of* Z
4. *Initial* $\mathbf{x}=(1/c_1,0,\dots,0)\in T\subset\mathbb{R}^n$

5. $\mathbf{g} = \mathbf{g}(\mathbf{x}) \in \mathbb{R}^n, \quad g_i(\mathbf{x}) = \frac{1}{2} \sum_{j,k=1}^{n} M(i,j,k) x_j x_k, \quad i \in [n]$

6. $H = H(\mathbf{x}) \in \mathbb{R}^{n \times n}, \quad H_{ij}(\mathbf{x}) = \sum_{k=1}^{n} M(i,j,k) x_k, \quad i,j \in [n]$

7. $V = V(\mathbf{x}) = \frac{1}{6} \sum_{i,j,k=1}^{n} M(i,j,k) x_i x_j x_k$

8. *Solve for* $\mathbf{d} \in \mathbb{R}^{n-1}$ *in* $A^T \left(H - \frac{2}{3} V^{-1} \mathbf{g}\mathbf{g}^T \right) A\mathbf{d} = -A^T \mathbf{g}$

9. *Compute* $V(A\mathbf{d})$

10. $t = \dfrac{-\mathbf{d}^T A^T H A \mathbf{d} - sign(V(A\mathbf{d}))\sqrt{(\mathbf{d}^T A^T H A \mathbf{d})^2 - 12 \mathbf{d}^T A^T \mathbf{g} V(A\mathbf{d})}}{6V(A\mathbf{d})}$

11. $\mathbf{x} = \mathbf{x} + tA\mathbf{d}$

12. *If* $t||A\mathbf{d}|| > \varepsilon$ *go to* 5.

If $\sum_{i=1}^{n} c_i Y_i$, the approximation of μ that is in use, is everywhere positive, the iteration presented above usually converges fast. If $\mathbf{x}^* \in \partial K$, the condition $\mu_{\mathbf{x}} \geqslant 0$ must be checked (e.g., by testing $\mu_{\mathbf{x}}(\omega)$ at randomly chosen points ω). Such an exception can occur when the solution set $\mathscr{R}$ has edges, corners or planar parts. However, the result is obtained in a convenient form as a support function. Using the formula (3.1) we can immediately compute the surface X parametrized in normal coordinates.

Other iteration methods can be used as well, of course. The simplest one, requiring no computation of $A, \mathbf{d}$, or H, is the gradient method. The projection $\mathbf{f}$ of the gradient $\mathbf{g}(\mathbf{x})$ of the volume $V(\mathbf{x})$ onto the plane $(\mathbf{x} - \mathbf{c})^T \mathbf{c} = 0$ (in which the volume is to be maximized) is

$$\mathbf{f} = \mathbf{g}(\mathbf{x}) - \frac{\mathbf{c}^T \mathbf{g}}{\mathbf{c}^T \mathbf{c}} \mathbf{c}.$$

The iteration always proceeds in this direction until the angle α between $\mathbf{c}$ and G, its cosine given by

$$\cos \alpha = \frac{\mathbf{c}^T \mathbf{g}}{||\mathbf{c}||\,||\mathbf{g}||},$$

is sufficiently small. The step length t for the updated value $\mathbf{x} = \mathbf{x} + t\mathbf{f}$ is given by the local maximum of the third-degree polynomial $\hat{V}(t) = V(\mathbf{x} + t\mathbf{f})$, for which we obtain

$$t = -\frac{W(\mathbf{x}) + \sqrt{W(\mathbf{x})^2 - 3V(\mathbf{f})W(\mathbf{x})}}{3V(\mathbf{f})},$$

where $W(\mathbf{x}) \equiv \mathbf{x}^T \mathbf{g}(\mathbf{f})$. The initial value for $\mathbf{x}$ can be, e.g., a ball in the iteration plane: $\mathbf{x}_0 = (c_1 + 1/c_1 \sum_{i=2}^{n} c_i^2, 0, \ldots, 0)$. If the final $\mathbf{x}$ is multiplied by $\sqrt{||\mathbf{c}||/||\mathbf{g}||}$, the size of the surface corresponds to that associated with the given $\mathbf{c}$.

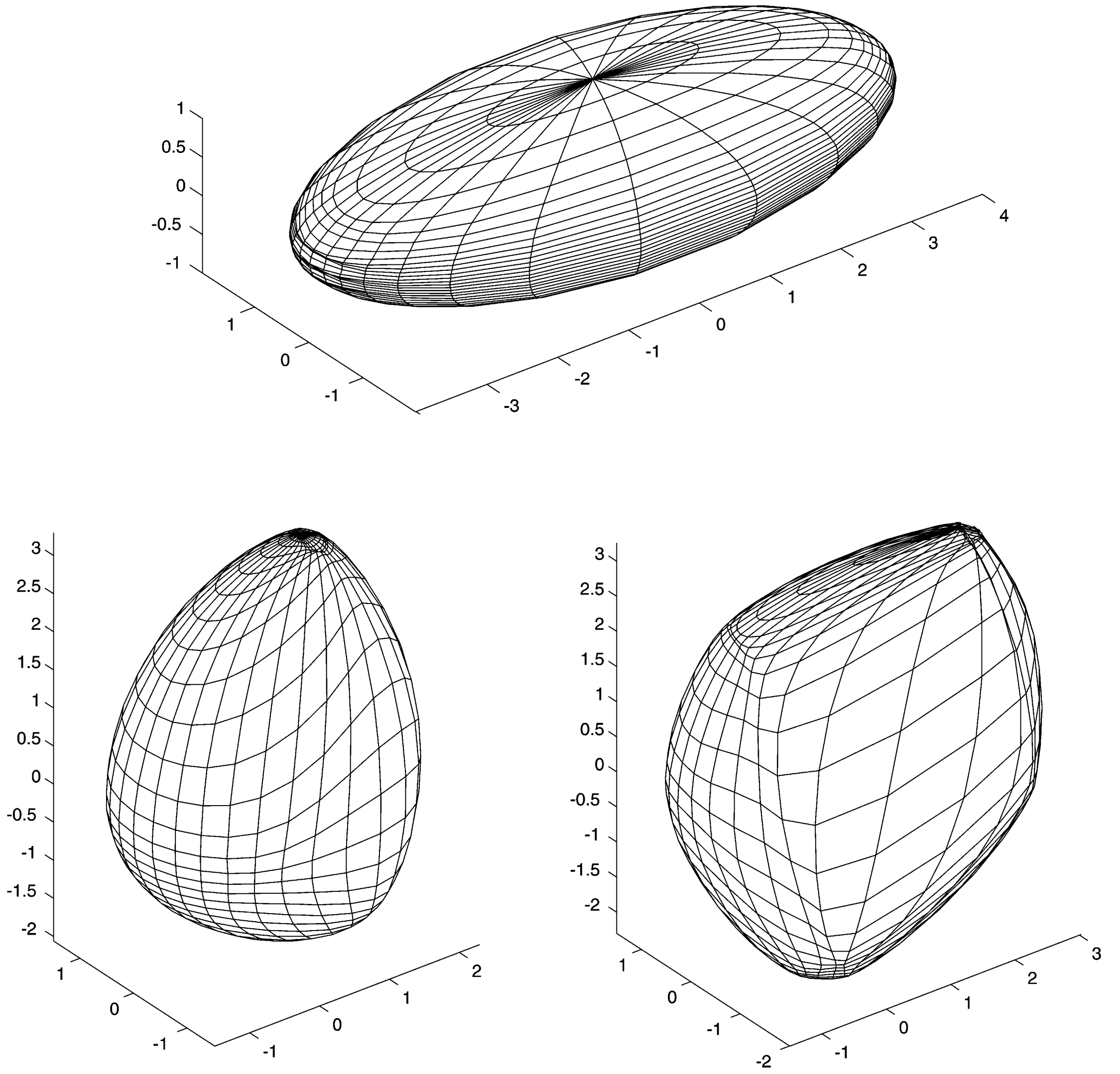

Fig. 1. Sample shapes.

Numerical tests indicate that the optimization in finite function space proceeds remarkably well. We produced arbitrary convex test shapes as in [3], i.e., by creating complex surfaces by adding together the support functions of simple but strongly curved convex primitives (such as analytically tractable 'droplets' or 'pins' formed by first rotating convex curves around z-axis and then tilting the resulting surface to a new position). In Fig. 1 we show some of the convex shapes used in our tests. The grid lines are isolatitudes and longitudes of the surface normal, equally spaced on the Gaussian sphere. The truncation order and degree for the spherical harmonics series of the curvature function μ and the support function ρ were typically 10 (resulting in 118 coefficients), well sufficient to depict shapes such as those in Fig. 1.

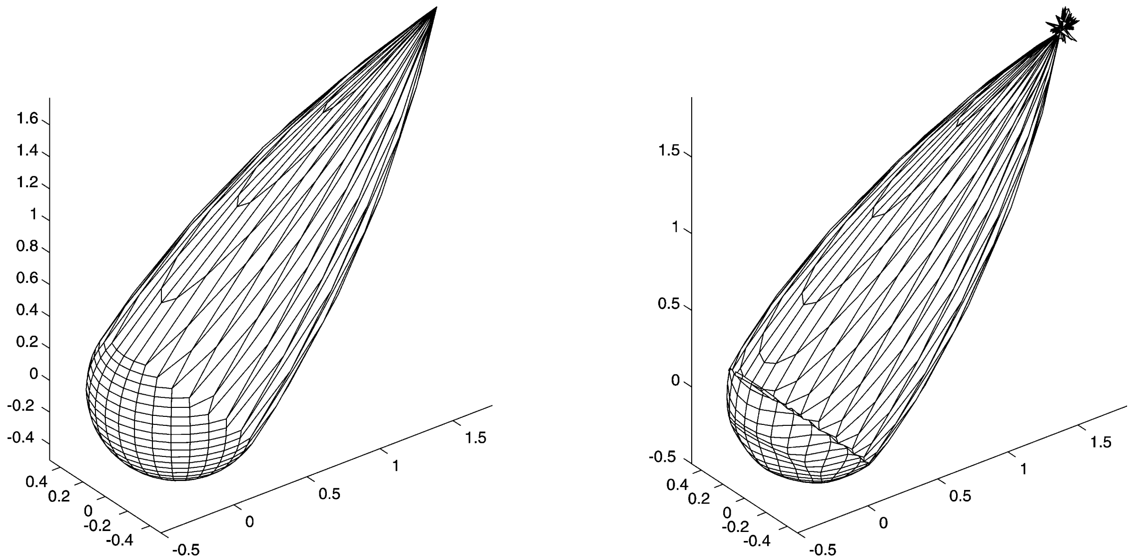

Fig. 2. A pathological sample shape and its reconstructed version.

The number of iterations required is typically of the same order or less than the number of parameters, depending on the complexity of the shape and the iteration method; Newton's method converges faster than the gradient method. The actual computation time is very short: the saturated result is obtained almost instantly. Sharp turns or large flat areas on the surface produced no difficulties when $\mu > 0$ (as given by the initial spherical harmonics series) as in Fig. 1. In such cases the results obtained were visually indistinguishable from the initial shapes. To get a more quantitative idea of the accuracy, we compared the optimized shape with the initial one using the convenient estimator $\Delta\rho$ introduced in [2]:

$$\Delta\rho = \frac{1}{4\pi}\int_{S^2} \frac{|\rho_{\rm in} - \rho_{\rm app}|}{\rho_{\rm in}}\,{\rm d}\omega,$$

where $\rho_{\rm in}$ and $\rho_{\rm app}$ are, respectively, the initial and the approximate support functions. The optimization procedure routinely obtains $\Delta\rho \approx 0.001$—values smaller than this are already in the region of numerical noise due to the fact that the initial truncated series of μ never corresponds exactly to the initial $\rho_{\rm in}$ from which it was computed.

An interesting fact is that the method is not limited to proper curvature functions. In Fig. 2a we show a droplet with a sharp top: the truncated spherical harmonics series of μ, approximating the actual μ that vanishes on a substantial section of the Gaussian sphere, necessarily reaches negative values at some normal directions ω near the top. Despite this, the optimization procedure finds a support function that produces the shape shown in Fig. 2b: the only errors in the result are the small 'fishtail' at the top and the flattening of the bottom. Thus, even if neither the original curvature function nor the final support function are legal, the surface drawn using (3.1) can still be quite acceptable.

Although the convergence of our method cannot be proved as rigorously as that for polyhedra in [5], we have found that optimization proceeds robustly and starts saturating at the correct answer. Thus, whenever the original curvature function can be well described with a spherical harmonics series, our method is clearly recommendable. If the polyhedron representation is more suitable, the method of [5] is, of course, quite as robust although slower due to geometric reconstructions (if there are hundreds or thousands of facets as is usually that case). The main advantage of using function space is its simplicity: constructing a polyhedron representation is always considerably harder (this is why the method of [5], specifically tailored to the polyhedron problem, is usually preferable over merely inserting the objective function in some standard optimization procedure as in [8,9]). For example, the shapes and numbers of vertices of individual facets may change considerably during iteration even if the global shape does not. Thus the reconstruction of a polyhedron may be seen locally unstable, whereas in function space the end result is stable at all scales.

4. Theorems and lemmas

We present here some properties of the "volume function" V and the "set of convex points" K.

Lemma 1. *$V(\mathbf{x}) > 0$ in the set K.*

Proof. Let $\mathbf{x} \in K$ and $V(\mathbf{x}) = 0$. Then $\mathscr{K}(\mathbf{x})$ is included in some plane W of $\mathbb{R}^3$, since it is convex and four points must be in the same plane. We can assume that $\rho_\mathbf{x}(\omega) \geqslant 0\ \forall \omega \in S^2$, since this is obtained by a translation without changing $\mu_\mathbf{x}$ and $V(\mathbf{x})$. We can rotate the coordinate system of $\mathbb{R}^3$ such that the new z-axis is in the plane W and in the new coordinate system

$$\rho_\mathbf{x}(0,\phi) = \begin{cases} a \sin\phi, & \text{if } 0 \leqslant \phi \leqslant \pi, \\ b|\sin\phi|, & \text{if } \pi < \phi < 2\pi, \end{cases} \qquad a, b \geqslant 0.$$

This is not a differentiable function at the point $\phi = 0$ except when $a = b = 0$. On the other hand, the support function $\rho_\mathbf{x}$ has the representation $\rho_\mathbf{x} = \sum_{k=1}^{n} y_k Y_k + \sum_{m=-1,0,1} z_m Y_1^m$ in the new coordinate system. Thus $a = b = 0$ and, since this is valid for every rotation described above, $\rho_X = 0$ and so $\mathbf{x} = 0$ contradicting the fact that $\mathbf{x}^T\mathbf{c} = 1$. □

Lemma 2. *$V^{1/3}$ is a strictly concave function on K.*

Proof. Let $\mathbf{x}, \mathbf{y} \in K$ and $0 < t < 1$. The convex set $\mathscr{K}(t\mathbf{x} + (1-t)\mathbf{y})$ contains the set with support function $t\rho_\mathbf{x} + (1-t)\rho_\mathbf{y}$: $t\mathscr{K}(\mathbf{x}) + (1-t)\mathscr{K}(\mathbf{y}) \subset \mathscr{K}(t\mathbf{x} + (1-t)\mathbf{y})$. According to the basic property (2.4) and Minkowski's Theorem B

$$\begin{aligned} V(t\mathbf{x} + (1-t)\mathbf{y}) &\geqslant V(t\mathscr{K}(\mathbf{x}) + (1-t)\mathscr{K}(\mathbf{y})) \\ &= t^3 V(\mathbf{x}) + 3t^2(1-t)V(\mathscr{K}(\mathbf{x}), \mathscr{K}(\mathbf{x}), \mathscr{K}(\mathbf{y})) \\ &\quad + 3t(1-t)^2 V(\mathscr{K}(\mathbf{x}), \mathscr{K}(\mathbf{y}), \mathscr{K}(\mathbf{y})) + (1-t)^3 V(\mathbf{y}) \end{aligned}$$

$$\begin{aligned}&\geqslant t^3V(\mathbf{x})+3t^2(1-t)V^{2/3}(\mathbf{x})V^{1/3}(\mathbf{y})+3t(1-t)^2V^{1/3}(\mathbf{x})V^{2/3}(\mathbf{y})\\&\quad+(1-t)^3V(\mathbf{y})+3t^2(1-t)(\kappa/d)(1-d)^6V^{2/3}(\mathbf{x})V^{1/3}(\mathbf{y})\\&\quad+3t(1-t)^2(\kappa/D^5)(D-1)^6V^{1/3}(\mathbf{x})V^{2/3}(\mathbf{y})\\&>(tV^{1/3}(\mathbf{x})+(1-t)V^{1/3}(\mathbf{y}))^3.\end{aligned}$$

Note that we have eliminated translations of sets and thus $D\neq 1$ or $d\neq 1$. Hence

$$V^{1/3}(t\mathbf{x}+(1-t)\mathbf{y})>tV^{1/3}(\mathbf{x})+(1-t)V^{1/3}(\mathbf{y})$$

which shows the strict concavity. □

Lemma 3. *The set K is bounded.*

Proof. Let $\mathbf{x}_0\in K$ be an interior point (in the topology of T), for instance $\mathbf{x}_0=(1/c_1,0,\dots,0)$. Then $T=\mathbf{x}_0+Z=\{\mathbf{x}_0+t\mathbf{e}\,|\,t\in\mathbb{R},\ \mathbf{e}\in Z,||\mathbf{e}||=1\}$. The function $V(\mathbf{x}_0+t\mathbf{e})$ of the variable t is a polynomial of degree 3 at most, and it is bounded in the set $N_{\mathbf{e}}=\{t\in\mathbb{R}\,|\,\mathbf{x}_0+t\mathbf{e}\in K\}$: after Minkowski $0<V(\mathbf{x}_0+t\mathbf{e})\leqslant V(\mathscr{R})$ in this set, where $\mathscr{R}$ corresponds to μ and $\langle\rho_{\mathscr{R}},\mu\rangle=1$. Thus either $N_{\mathbf{e}}$ is bounded or $V(\mathbf{x}_0+t\mathbf{e})$ is a constant: the latter contradicts the fact that $V^{1/3}$ is strictly concave in K. Thus $N_{\mathbf{e}}$ is bounded for every $\mathbf{e}\in Z$.

As a consequence, the function $t(\mathbf{e}):=\sup\{t\in\mathbb{R}\,|\,\mathbf{x}_0+t\mathbf{e}\in K\}$ is a finite on the unit sphere $S=\{\mathbf{e}\in Z\,|\,||\mathbf{e}||=1\}$. For every $\mathbf{e}_0$ we can find $\omega\in S^2$ such that $\mu_{\mathbf{x}_0+(t(\mathbf{e}_0)+1)\mathbf{e}_0}(\omega)<0$. Since $\mu_{\mathbf{x}_0+t\mathbf{e}}(\omega)$ is continuous in $\mathbf{e}$, we can find an open neighbourhood $U(\mathbf{e}_0)$ in S such that $\mu_{\mathbf{x}_0+(t(\mathbf{e}_0)+1)\mathbf{e}}(\omega)<0$, when $\mathbf{e}\in U(\mathbf{e}_0)$. Thus $t(\mathbf{e})<t(\mathbf{e}_0)+1$ in $U(\mathbf{e}_0)$ (K is convex). Since S is compact, $t_0:=\sup\{t(\mathbf{e})\,|\,\mathbf{e}\in S\}<\infty$. Let $\mathbf{x}\in K$. Then $\mathbf{x}=\mathbf{x}_0+t\mathbf{e}$, where $\mathbf{e}\in S$ and $t\geqslant 0$. Thus $||\mathbf{x}-\mathbf{x}_0||=t\leqslant t(\mathbf{e})\leqslant t_0$. □

Theorem 1. *The function V has a unique maximum point in the set K.*

Proof. Since K is a compact set, V obtains its maximum in K, and since $V^{1/3}$ is strictly concave in K (which is convex), the maximum point is unique. □

Theorem 2. *Let the curvature function $\mu=\mu_{\mathscr{R}}$ be an everywhere positive C^∞-function and let $\mathbf{x}_n^*\in K_n$ be the maximum point defined by Theorem 1 corresponding to the dimension index $n\in\mathbf{N}_+$. Then*

$$\lim_{n\to\infty}\sup_{\omega\in S^2}|\rho_{\mathbf{x}_n^*}(\omega)-\rho(\omega)|=0,$$

where $\rho=(3V_{\mathscr{R}})^{-1}\rho_{\mathscr{R}}$.

Proof. As Nirenberg has shown, under these assumptions ρ is a C^∞-function [11]. Let $\rho=\sum_{i=1}^\infty y_iY_i$; this series converges uniformly in S^2 and $\sum_{i=1}^\infty i^py_i^2<\infty$ for every $p\in\mathbf{N}$. Hence the series

$$\sum_{i=1}^\infty y_i\left(\frac{\partial^l\partial^k}{\partial\phi^l\partial\theta^k}Y_i\right)$$

converges uniformly for every $k, l \in \mathbf{N}$ and

$$\frac{\partial^l \partial^k}{\partial \phi^l \partial \theta^k}\rho = \sum_{i=1}^{\infty} y_i \left(\frac{\partial^l \partial^k}{\partial \phi^l \partial \theta^k} Y_i \right)$$

(note recursion formulas of derivatives of Legendre functions [1]). Denote $\mathbf{y}_n = (y_1, \ldots, y_n)$; $\rho_{\mathbf{y}_n} = \sum_{i=1}^{n} y_i Y_i$. Then $\mu_{\mathbf{y}_n} = A(\rho_{\mathbf{y}_n})C(\rho_{\mathbf{y}_n}) - B(\rho_{\mathbf{y}_n})^2$ converges uniformly to $\mu_\rho = A(\rho)C(\rho) - B(\rho)^2 = (3V_{\mathscr{R}})^{-2}\mu_{\mathscr{R}}$ implicating that $\rho_{\mathbf{y}_n}$ is a support function, when $n \geqslant n_0$ for some $n_0 \in \mathbf{N}_+$. Also $\int_{S^2} \rho_{\mathbf{y}_n} \, \mathrm{d}\mu \xrightarrow{n} \int_{S^2} \rho \, \mathrm{d}\mu = 1$. By normalizing $\mathbf{y}_n$ we get a sequence $(\mathbf{z}_n)_1^\infty$, $\mathbf{z}_n \in K_n$ when $n \geqslant n_0$, with the property that $V(\mathbf{z}_n) \xrightarrow{n} V(\rho) = (3V_{\mathscr{R}})^{-3} V_{\mathscr{R}}$. Since $V(\mathbf{z}_n) \leqslant V(\mathbf{x}_n^*) \leqslant V(\rho)$ when $n \geqslant n_0$, $V(\mathbf{x}_n^*) \xrightarrow{n} V(\rho)$ and using Minkowski's Theorem B yields the result $\lim_{n\to\infty} \sup_{\omega \in S^2} |\rho_{\mathbf{x}_n^*}(\omega) - \rho(\omega)| = 0$. □

Lemma 4. $\nabla V(\mathbf{x}) = \mathbf{c}(\mathbf{x})$ *and* $\frac{\partial^2 V}{\partial x_i \partial x_j}(\mathbf{x}) = \sum_{k=1}^{n} M(i,j,k) x_k$, $i, j \in [n]$.

Proof. Since the coefficients $M(i,j,k)$ are translation invariant,

$$\begin{aligned} V(\mathbf{x}) &= \frac{1}{6} \sum_{i,j,k=1}^{n} M(i,j,k) x_i x_j x_k \\ &= \frac{1}{6} \sum_{i=1}^{n} M(i,i,i) x_i^3 + \frac{1}{2} \sum_{i,j; i \neq j} M(i,i,j) x_i^2 x_j + \frac{1}{6} \sum_{i,j,k; i\neq j, i\neq k, j\neq k} M(i,j,k) x_i x_j x_k \end{aligned}$$

and so

$$\begin{aligned} \frac{\partial V}{\partial x_s}(\mathbf{x}) &= \frac{1}{2} M(s,s,s) x_s^2 + \sum_{j \neq s} M(s,s,j) x_s x_j + \frac{1}{2} \sum_{i \neq s} M(i,i,s) x_i^2 \\ &\quad + \frac{1}{2} \sum_{j,k; s\neq j, s\neq k, j\neq k} M(s,j,k) x_j x_k \\ &= \frac{1}{2} \sum_{i,j} M(s,i,j) x_i x_j = c_s(\mathbf{x}), \quad s \in [n], \end{aligned}$$

which proves the first statement. The second part can be shown similarly. □

Theorem 3. *The function* V *(as well as* $V^{1/3}$*) has at most one* T*-constrained stationary point in* K*. When it exists, it is a strict maximum point and thus* $\mathbf{x}^*$.

Proof. If $\bar{\mathbf{x}} \in T$ is a T-constrained stationary point of V, the vector $\mathbf{c}(\bar{\mathbf{x}}) = \nabla V(\bar{\mathbf{x}})$ is $\mathbf{c}$ (except for normalization), since $A^T \nabla V(\bar{\mathbf{x}}) = 0$. We will now work on the surface $V(\mathbf{x}) = 1$. We denote the unique radial counterpoint of $\mathbf{x} \in K \subset T$ on the surface $V(\mathbf{x}) = 1$ by $\mathbf{y}$ and the image of K in this radial mapping by $\bar{K}$. Thus, $V(\mathbf{x}) = (\mathbf{y}^T\mathbf{c})^{-3}$ and $\nabla V(\mathbf{x}) = \mathbf{c}(\mathbf{x}) = (\mathbf{y}^T\mathbf{c})^{-2}\mathbf{c}(\mathbf{y}) = (\mathbf{y}^T\mathbf{c})^{-2}\nabla V(\mathbf{y})$. Let $\bar{\mathbf{y}}$ be the radial counterpoint of the stationary point $\bar{\mathbf{x}} \in K$: $\nabla V(\bar{\mathbf{y}}) = \lambda \mathbf{c}$. On the other hand $\nabla V(\bar{\mathbf{y}}) = \mathbf{c}(\bar{\mathbf{y}})$ and thus $(\nabla V(\bar{\mathbf{y}}))^T \bar{\mathbf{y}} = \mathbf{c}(\bar{\mathbf{y}})^T \bar{\mathbf{y}} = 3V(\bar{\mathbf{y}}) = 3 > 0$. According to Minkowski $\bar{\mathbf{y}}^T \mathbf{c} = \langle \rho_{\bar{\mathbf{y}}}, \mu \rangle = \int_{S^2} \rho_{\bar{\mathbf{y}}} \, \mathrm{d}\mu > 0$. Hence

$$\lambda = \frac{(\nabla V(\bar{\mathbf{y}}))^T \bar{\mathbf{y}}}{\bar{\mathbf{y}}^T \mathbf{c}} > 0.$$

If $\mathbf{y} \in \bar{K}$, Minkowski's theorem gives

$$\begin{aligned}\mathbf{y}^T\mathbf{c} &= \frac{1}{\lambda}\mathbf{y}^T\nabla V(\bar{\mathbf{y}}) = \frac{1}{\lambda}\mathbf{y}^T\mathbf{c}(\bar{\mathbf{y}}) = \frac{1}{\lambda}\langle \rho_{\mathbf{y}}, \mu_{\bar{\mathbf{y}}}\rangle \\ &\geqslant \frac{1}{\lambda}\langle \rho_{\bar{\mathbf{y}}}, \mu_{\bar{\mathbf{y}}}\rangle = \frac{1}{\lambda}\bar{\mathbf{y}}^T\mathbf{c}(\bar{\mathbf{y}}) = \frac{1}{\lambda}\bar{\mathbf{y}}^T\nabla V(\bar{\mathbf{y}}) = \bar{\mathbf{y}}^T\mathbf{c}.\end{aligned}$$

The equal sign holds true only when the convex sets corresponding to $\mathbf{y}$ and $\bar{\mathbf{y}}$ are the same except for a translation. However, we have chosen the basis (Y_i) such that translations have been eliminated. Thus if $\mathbf{y}^T\mathbf{c} = \bar{\mathbf{y}}^T\mathbf{c}$, then $\mathbf{y} = \bar{\mathbf{y}}$. Hence $\mathbf{y}$ is the only "stationary point" in $\bar{K}$, and it is also a strict minimum of the inner product. The counterpoint $\bar{\mathbf{x}}$ is the only T-constrained stationary point in K and it is a strict maximum point of V on K, so $\bar{\mathbf{x}} = \mathbf{x}^*$. □

Theorem 4. *$\tilde{H}_p$ is negative definite in the interior of K.*

Proof. The function $V^{1/3}$ is strictly concave in K, which implies that $\tilde{H}_p(\mathbf{x})$ is negative semidefinite in K. Let $\mathbf{x} \in K$ be an interior point and let $\mathbf{e} \in Z$. It is sufficient to prove that $\mathbf{e}^T\tilde{H}(\mathbf{x})\mathbf{e} \neq 0$. We use the Taylor series of the cubic root (t is small)

$$\begin{aligned}V^{1/3}(\mathbf{x}+t\mathbf{e}) &= (V(\mathbf{x}) + \mathbf{e}^T\mathbf{g}(\mathbf{x})t + 1/2\mathbf{e}^TH(\mathbf{x})\mathbf{e}t^2 + V(\mathbf{e})t^3)^{1/3} \\ &= V^{1/3}(\mathbf{x})[1 + 1/3(\mathbf{e}^T\mathbf{g}(\mathbf{x})t + 1/2\mathbf{e}^TH(\mathbf{x})\mathbf{e}t^2 + V(\mathbf{e})t^3)/V(\mathbf{x}) \\ &\quad - 1/9(\mathbf{e}^T\mathbf{g}(\mathbf{x})t + 1/2\mathbf{e}^TH(\mathbf{x})\mathbf{e}t^2 + V(\mathbf{e})t^3)^2/V(\mathbf{x})^2 + 5/81\cdots] \\ &= \varepsilon(t)t^3 + V^{1/3}(\mathbf{x}) \\ &\quad + 1/3V(\mathbf{x})^{-2/3}\mathbf{e}^T\mathbf{g}(\mathbf{x})t + (1/6V(\mathbf{x})^{-2/3}\mathbf{e}^TH(\mathbf{x})\mathbf{e} - 1/9V(\mathbf{x})^{-5/3}\mathbf{e}^T\mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})^T\mathbf{e})t^2 \\ &\quad + (1/3V(\mathbf{x})^{-2/3}V(\mathbf{e}) - 1/9V(\mathbf{x})^{-5/3}\mathbf{e}^T\mathbf{g}(\mathbf{x})\mathbf{e}^TH(\mathbf{x})\mathbf{e} + 5/81V(\mathbf{x})^{-8/3}(\mathbf{e}^T\mathbf{g}(\mathbf{x}))^3)t^3.\end{aligned}$$

Because of concavity

$$V^{1/3}(\mathbf{x}+t\mathbf{e}) - V^{1/3}(\mathbf{x}) - 1/3V(\mathbf{x})^{-2/3}\mathbf{e}^T\mathbf{g}(\mathbf{x})t = V^{1/3}(\mathbf{x}+t\mathbf{e}) - V^{1/3}(\mathbf{x}) - \mathbf{e}^T\tilde{\mathbf{g}}(\mathbf{x})t \leqslant 0.$$

If $\mathbf{e}^T\tilde{H}(\mathbf{x})\mathbf{e} = 0$, also the coefficient of t^3 must be zero: we can replace $\mathbf{e}$ by $-\mathbf{e}$. Hence $\mathbf{e}^TH(\mathbf{x})\mathbf{e} = 2/3V(\mathbf{x})^{-1}(\mathbf{e}^T\mathbf{g}(\mathbf{x}))^2$ and $V(\mathbf{e}) = 1/27V(\mathbf{x})^{-2}(\mathbf{e}^T\mathbf{g}(\mathbf{x}))^3$. Thus

$$\begin{aligned}V^{1/3}(\mathbf{x}+t\mathbf{e}) &= V^{1/3}(\mathbf{x})[1 + 3(1/3V(\mathbf{x})^{-1}\mathbf{e}^T\mathbf{g}(\mathbf{x}))t + 3(1/3V(\mathbf{x})^{-1}\mathbf{e}^T\mathbf{g}(\mathbf{x}))^2t^2 \\ &\quad + (1/3V(\mathbf{x})^{-1}\mathbf{e}^T\mathbf{g}(\mathbf{x}))^3t^3]^{1/3} \\ &= V^{1/3}(\mathbf{x}) + \mathbf{e}^T\tilde{\mathbf{g}}(\mathbf{x})t\end{aligned}$$

which contradicts the strict concavity. Hence $\mathbf{e}^T\tilde{H}(\mathbf{x})\mathbf{e} < 0$ and the projected Hessian $\tilde{H}_p(\mathbf{x})$ is negative definite. □

References

[1] G. Arfken, Mathematical Methods for Physicists, Academic Press, New York, 1968.
[2] M. Kaasalainen, L. Lamberg, K. Lumme, E. Bowell, Interpretation of lightcurves of atmosphereless bodies. 1. General theory and new inversion schemes, Astron. Astrophys. 259 (1992) 318–332.
[3] M. Kaasalainen, L. Lamberg, K. Lumme, Interpretation of lightcurves of atmosphereless bodies. 2. Practical aspects of inversion, Astron. Astrophys. 259 (1992) 333–340.
[4] M. Kaasalainen, J. Torppa, Optimization methods for asteroid lightcurve inversion. I. Shape determination, Icarus (2001), in press.
[5] L. Lamberg, On the Minkowski problem and the lightcurve operator, Academia Scientiarum Fennica, Series A. I. Mathematica dissertationes, 1993.
[6] K. Leichtweiss, Konvexe Mengen, Springer, Berlin, 1980.
[7] H. Lewy, On differential geometry in the large 1 (Minkowski's problem), Trans. Amer. Math. Soc. 43 (2) (1938) 258–270.
[8] J.J. Little, An iterative method for reconstructing convex polyhedra from extended Gaussian images, in: Proceedings of the National Conference on Artificial Intelligence Washington D.C., 1983, pp. 247–250.
[9] J.J. Little, Recovering shape and determining attitude from extended Gaussian images, Technical Report TN 85-2 at the University of British Columbia, Vancouver, 1985.
[10] H. Minkowski, Volumen und Oberfläche, Math. Ann. 57 (1903) 447–495.
[11] L. Nirenberg, The Weyl and Minkowski problems in differential geometry in the large, Commun. Pure Appl. Math. 6 (1953) 337–394.