17.5 Theorem Assume $m > 0$ is a constant.

i) If

$$\tau \leq \frac{\lambda_0(D)}{m+1}$$

then τ is not a TE

ii) If

$$\frac{(1+\frac{m}{2})^2}{1+m} \geq \frac{\mu_p}{\lambda_0},$$

then $\exists$ (p+1) TE's τ with

$$\tau \leq \frac{m+2}{m+1} \frac{\lambda_0}{2}$$

Proof i) Let $n = m+1$. From

$$T_\tau = (\Delta+\tau n) \frac{1}{m} (\Delta+\tau n) - \tau(\Delta+\tau n)$$

it follows

$$m\, t_\tau(u) = \|(\Delta+\tau n)u\|^2 - m\tau \int u(\Delta+\tau n)u$$

$$> m\tau \left[\|\nabla u\|^2 - n\tau \|u\|^2\right]$$

$$\geq m\tau \left[\lambda_0 - n\tau\right] \|u\|^2 .$$

Thus t_τ is positive def. $\Rightarrow$ τ is not TE.

ii) By Lemma ?? we need to prove

$$t_\tau(u) \leq 0$$

for $u \in V_{p+1}$, $\dim V_{p+1} \geq p+1$.

Assume $\|u\| = 1$.

We use

$$T_\tau = \Delta \frac{1}{m} \Delta + \tau\left(\Delta \frac{1}{m} + \left(1+\frac{1}{m}\right)\Delta\right) + \tau^2\left(1+\frac{1}{m}\right)$$

$$= \frac{1}{m}\Delta^2 + 2\frac{\tau}{m}\Delta + \tau\Delta + \tau^2\left(1+\frac{1}{m}\right)$$

$\Rightarrow$

$$m\, t_\tau(u) = \|\Delta u\|^2 + n\tau^2 - 2\left(1+\frac{m}{2}\right)\tau \|\nabla u\|^2$$

$$\leq n\tau^2 - 2\left(1+\frac{m}{2}\right)\tau\lambda_p + \|\Delta u\|^2$$

If $\tau = \frac{1+\frac{m}{2}}{n}\lambda_0 = \frac{1+\frac{m}{2}}{1+m}\lambda_0$, we get

$$m\, t_\tau(u) \leq -\frac{\left(1+\frac{m}{2}\right)^2}{1+m}\lambda_0^2 + \|\Delta u\|^2$$

If we assume further u to space V_{p+1} spanned by $u_1, \ldots, u_{p+1} \in H_0^2(D)$

$$\Delta^2 u_i = \mu_i u_i$$

we get

$$m\, t_\tau(u) \leq -\frac{\left(1+\frac{m}{2}\right)^2}{1+m}\lambda_0^2 + \mu_p^2 \leq 0$$

by hypothesis.

∎

Theorem 6 Assume $m \in C^2(\bar{D})$, $m > \delta > 0$.

i) If $\tau \leq \frac{\lambda_0}{\|m\|_\infty + 1}$

then τ is not a TE.

ii) If

$$\inf m \geq 4\left(\frac{\mu_p^{1/2}}{\lambda_0} + \frac{\mu_p}{\lambda_0}\right).$$

then $\exists$ $(p+1)$ TE's τ with

$$\tau \leq \frac{\lambda_0}{2}\left(\frac{\inf m - 2\sqrt{\frac{\mu_p^2}{\lambda_0}}}{\inf m + 1}\right)$$

Proof i) Since $m > 0$

$$t_\tau(u) = \int \frac{1}{m}\|(\Delta + \tau n)u\|^2 - \tau\int \bar{u}(\Delta + \tau n)u$$

$$\geq \tau\|\nabla u\|^2 - \tau^2\int n|u|^2$$

$$\geq \tau\left(\lambda_0 - \tau(1+\|m\|_\infty)\right)\|u\|^2 > 0$$

ii) We write

$$t_\tau(u) \overset{(5.4)}{=} \tau^2\int\left(1+\frac{1}{m}\right)|u|^2 - \tau\left(\|\nabla u\|^2 - \int\frac{1}{m}(\bar{u}\Delta u + u\Delta\bar{u})\right)$$

$$+ \int\frac{1}{m}|\Delta u|^2$$

Assume $\|u\| = 1$ and denote $S = \|\frac{1}{m}\|_\infty = \frac{1}{\inf m}$

to see that

$$t_\tau(u) \leq \tau^2(1+S) - \tau\left(\|\nabla u\|^2 - 2S\|\Delta u\|\right) + S\|\Delta u\|^2$$

Restricting to V_p $\Rightarrow$

$$t_\tau(u) \leq \tau^2(1+S) - \tau\left(\lambda_0 - 2S\sqrt{\mu_p}\right) - S\mu_p$$

The r.h.s. is minimized if

$$z = \frac{\lambda_0 - 2S\sqrt{\mu_p}}{2(1+S)}$$

$$\Rightarrow \quad l_\tau(u) \leq -\frac{(\lambda_0 - 2S\sqrt{\mu_p})^2}{4(1+S)} + S\mu_p$$

Writing $A = \frac{\lambda_0}{\sqrt{\mu_p}}$ we have

$$l_\tau(u) \leq 0 \iff$$

$$\left(\frac{\lambda_0}{\sqrt{\mu_p}} - 2S\right)^2 \cancel{\mu_p} + 4(1+S)S\cancel{\mu_p} \leq 0$$

$$\iff (A - 2S)^2 + 4S(1+S) \leq 0$$

$$\iff A^2 - 4(A+1)S \leq 0$$

$$\iff S \leq \frac{A^2}{4(A+1)}$$

$$\iff \inf(m) \geq \frac{4}{A} + \frac{4}{A^2} = 4\left(\frac{\sqrt{\mu_p}}{\lambda_0} + \frac{\mu_p}{\lambda_0^2}\right)$$

18. Scaling invariant spaces in scattering

Some speculation: (Jan 71)

If

$$u'' + k^2 u = k^2 f$$

$$\Rightarrow \quad u(s) = k^2 \int_{-\infty}^{\infty} \frac{e^{ik|s-t|}}{2ik} f(t)\, dt$$

Especially

$$(1) \qquad \|u\|_{L^\infty} \le \frac{|k|}{2} \|f\|_{L^1}$$

We want to find norms that would define dual spaces and that would scale linearly with dilatation and that the resolvent would be bounded similar to (1).

If $(\Delta + k^2) u = f$, set

$$v(x) = u(x/k) \quad \text{and} \quad g(x) = f(x/k)$$

then

$$(\Delta + 1) v = g$$

The B and B^*, L^2_δ and $L^2_{-\delta}$ don't scale:

$$(2) \qquad \|v\|_{L^2_{-\delta}} \le \begin{cases} \|f\|_{L^2_\delta} & \text{for small } k \\ k^{2\delta} \|f\|_{L^2_\delta} & \text{for large } k \end{cases}$$

Similar for B and B^*.

L^p estimates dilate and rotate

(3) $\|u\|_p \le C k^{n(1/q - 1/p)} \|f\|_q$, $\frac{1}{q} - \frac{1}{p} \ge \frac{2}{n+1}$

Cf. Kenig, Ruiz, Sogge 1987.

For large k estimate (2) is stronger than (3).

Goal: Find optimal scaling invariant spaces.

• Note: both L^2_δ and B fix the origin. This is not natural in scattering!

18.1 Definition Assume $\theta \in S^{n-1}$. We define

$$\|u\|_{\theta^{\infty,2}} = \sup_{\tau \in \mathbb{R}} \Big(\int_{x\cdot\theta = \tau} |u|^2 \, dS \Big)^{1/2}$$

$$\|u\|_{\theta^{1,2}} = \int_{\mathbb{R}} \Big(\int_{x\cdot\theta = \tau} |u|^2 \, dS \Big)^{1/2} d\tau$$

for measurable u.

$\Theta^{1,2}$ and $\Theta^{\infty,2}$ are the corresponding spaces.

<u>Remark</u> Analogously one can define spaces

$$\Theta^{p,q} \quad , \quad p,q \in [1,\infty].$$

C_0^∞ is dense in $\Theta^{1,2}$ (and in $\Theta^{p,q}$ for $1 < p,q < \infty$) but not in $\Theta^{\infty,2}$.

<u>18.2 Lemma</u>

i) $\| u(\cdot/k) \|_{\Theta^{\infty,2}} = |k|^{\frac{n-1}{2}} \| u \|_{\Theta^{\infty,2}}$

ii) $\| u(\cdot/k) \|_{\Theta^{1,2}} = |k|^{\frac{n+1}{2}} \| u \|_{\Theta^{1,2}}$

<u>Proof</u> i)
$$\| u(\cdot/k) \|^2_{\Theta^{\infty,2}} = \sup_t \int_{x\cdot\theta = t} |u(x/k)|^2 \, dS$$

Without loss of generality we may assume $\theta = e_1$. Now with $x' = (x_2, \ldots, x_n)$ we have

$$\| u(\cdot/k) \|^2_{\Theta^{\infty,2}} = \sup_{x_1} \int_{\mathbb{R}^{n-1}} |u(\tfrac{x_1}{k}, \tfrac{x'}{k})|^2 \, dx'$$

$$\overset{x'/k = y'}{=} \sup_{y_1} \int_{\mathbb{R}^{n-1}} |u(y_1, y')|^2 k^{n-1} \, dy'$$

$$= \| u \|^2_{\Theta^{\infty,2}}$$

ii) is ex. □

18.3 Definition Let $\{\Theta_j\}_{j=1}^{n}$ be ON basis for $\mathbb{R}^n$ and $\Theta_0 = \frac{1}{\sqrt{2}}(\Theta_1 + \Theta_2)$

We denote

$$\mathcal{J}_n = \{\Theta_j\}_{j=0}^{n+1}$$

Note that if we can show

$$\| R_0(\lambda + i0) f \|_{\Theta^{\infty,2}} \leq C(\lambda) \|f\|_{\Theta^{1,2}}$$

then Lemma 18.2 gives

(R) $$\| R(\lambda + i0) f \|_{\Theta^{\infty,2}} \leq \frac{C(1)}{k} \|f\|_{\Theta^{1,2}}$$

and we have linear scaling.

We will show below a stronger result

18.4 Theorem Assume $f \in C_0^\infty$ and $z \in \overline{\mathbb{C}}_+$.

Then for $u = R_0(z) f$ we have

(1) $$u = \sum_{i=1}^{n} u_i \quad , \text{ with}$$

(2) $$\| u_i \|_{\Theta_i^{\infty,2}} \leq \frac{C(n)}{k} \|f\|_{\Theta_i^{1,2}} \quad , \; i = 0, \ldots, n.$$

Moreover for ∇u we have

(3) $$\nabla u = \sum_{i=1}^{n} w_i$$

(4) $\|\nabla w_i\|_{\Theta_i^{\infty,2}} \leq C(n)\,\|f\|_{\Theta_i^{1,2}}$ and

(5) $\|\Delta u\|_{\Theta_i^{\infty,2}} \leq C(n)\left(\|f\|_{\Theta_i^{\infty,2}} + |k|\,\|f\|_{\Theta_i^{1,2}}\right)$

<u>Remark</u> Theorem 14.1.2 in [Hö] gives

(6) $\|u\|_{\Theta^{1,2}} \leq \sqrt{2}\,\|u\|_{B}$ and

(7) $\|u\|_{B^*} \leq \sqrt{2}\,\|u\|_{\Theta^{\infty,2}}$

Hence (R) and ~~sharpens~~ Theorem 15.4 Hörmanders resolvent estimate

$$R(\lambda + i0) : B \to B^* .$$

We next compare $\Theta^{p,2}$ norms for $p=1$ and $p=\infty$ with L^2_δ norms.

<u>18.5 Lemma</u> For $\delta > 1/2$, $f \in L^2_\delta$ and $u \in \Theta^{\infty,2}$ we have

i) $\|f\|_{\Theta^{1,2}} \leq \left(\dfrac{2\delta}{2\delta-1}\right)^{1/2} \|f\|_{L^2_\delta}$ and

ii) $\|u\|_{L^2_{-\delta}} \leq \left(\dfrac{2\delta}{2\delta-1}\right)^{1/2} \|u\|_{\Theta^{\infty,2}}$

Proof. Let $\theta \in S^{n-1}$. For any $x \in \mathbb{R}^n$ write

$$X = t\theta + y \quad , \text{ with } y\cdot\theta = 0$$

Now

$$\|f\|_{\theta,1,2} = \int\left(\int |f(t,y)|^2 dy\right)^{1/2} (1+t^2)^{\delta/2} (1+t^2)^{-\delta/2}\, dt$$

$$\leq \underbrace{\left(\int\int |f(t,y)|^2 (1+t^2)^{\delta/2}\right)^{1/2}}_{\leq \|f\|_{L^2_\delta}} \underbrace{\left(\int (1+t^2)^{-\delta}\, dt\right)^{1/2}}_{\leq \left(\frac{2\delta}{\delta-1}\right)^{1/2}}$$

□

18.7 Lemma Let $D \subset \mathbb{R}^n$ be a bounded domain and define

$$d(D,\theta) = \sup_{x\in D} \left| \{ t\in\mathbb{R} \mid x + t\theta \in D \} \right|$$

(Lebesgue measure)

$$d(D) = \sup_{\theta\in S^{n-1}} d(D,\theta)$$

Then $\forall \theta$

$$\left(\frac{1}{d(D,\theta)}\right)^{1/2} \|f\|_{\theta,1,2(D)} \leq \|f\|_{L^2(D)} \leq d(D,\theta)^{1/2} \|f\|_{\theta,2,\infty}$$

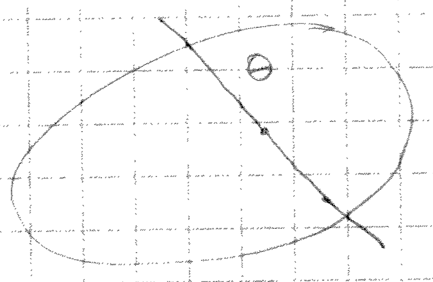
18.6 Lemma The constant $d(D)$ satisfies

i) $d(D) \leq \delta(D)$ = diameter of D

ii) If $D = \bigcup D_i$, then $d \leq \sum d(D_i)$

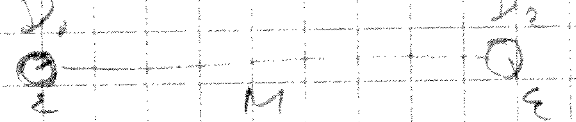
Proof i) is obvious.

ii) Same as written
sub-additivity [illegible]



□

Remark



$D = D_1 \cup D_2$

$\Rightarrow \delta(D) = M + \varepsilon$ and $d(D) = 2\varepsilon$