Def 18.1 $\theta \in S^{n-1}$, $D \subset \mathbb{R}^n$ domain

$$\|u\|_{\theta^\infty(D)} = \sup_{z \in \mathbb{R}} \int_{D \cap \{x\cdot\theta = z\}} |u|^2 \, dS$$

$$\|u\|_{\theta^1(D)} = \int_{z \in \mathbb{R}} \left(\int_{D \cap \{x\cdot\theta = z\}} |u|^2 \, dS \right)^{1/2} dz$$

Then

$$\|u(x/h)\|_{\theta^\infty} = h^{\frac{n-1}{2}} \|u\|_{\theta^\infty} , \quad h > 0$$

$$\|u(x/h)\|_{\theta^1} = h^{\frac{n+1}{2}} \|u\|_{\theta^1}$$

Def 18.3 $J_n = \{\theta_j\}_{j=0}^{n}$, $\theta_0 = \dfrac{\theta_1 + \theta_n}{\sqrt{2}}$, where $\{\theta_1, \dots, \theta_n\}$ is orthonormal basis.

Theorem 18.4 M For $k \in \mathbb{C}_+$, $f \in C_0^\infty$ and $u = R_0(k+i0)f$ $\exists$ $C = C(n) > 0$ s.t.

$$u = \sum_{i=0}^{n} u_i$$

a) $$\|u_i\|_{\theta_i^\infty} \le \frac{C(n)}{|k|} \|f\|_{\theta_i^1}$$

b) $$\nabla u = \sum_{i=0}^{n} w_i$$

c) $$\|w_i\|_{\theta_i^\infty} \le C(n) \|f\|_{\theta_i^1}$$

d) $$\|\Delta u_i\|_{\theta_i^\infty} \le C(n) \left(\|f\|_{\theta^\infty} + |k| \, \|f\|_{\theta_i^1} \right)$$

18.5

Lemma A For $\delta > 1/2$

$$\|f\|_{\Theta^1} \le \sqrt{\frac{2\delta}{2\delta-1}}\, \|f\|_{L^2_\delta}$$

$$\|u\|_{L^2_{-\delta}} \le \sqrt{\frac{2\delta}{2\delta-1}}\, \|u\|_{\Theta^\infty}$$

$$d(D,\theta) = \sup_{x\in D} \left| \{t \mid x+t\theta \in D\} \right|$$

$$d(D) = \sup_{\theta} d(D,\theta)$$

Note $\quad d(D) = \sup_{\ell} |\ell \cap D| \quad , \ \ell$ is a line in $\mathbb{R}^n$

Here $|\cdot|$ denotes 1-D Lebesgue measure.

18.6

Lemma B

$$\sqrt{\frac{1}{d(D,\theta)}}\, \|f\|_{\Theta^1(D)} \le \|f\|_{L^2(D)} \le \sqrt{d(D,\theta)}\, \|f\|_{\Theta^\infty(D)}$$

$$D = \cup D_i \implies d(D) \le \sum d(D_i)$$

18.7

Lemma C $\|f\|_{\Theta^1} \le \|f\|_{B}$

$$\|u\|_{B^*} \le \|u\|_{\Theta^\infty}.$$

Hormander Theorem 14.1.2.

<u>Corollary</u> 18.8 For $\delta > 1/2$

i) $\|u\|^2_{L_{-\delta}} \le \frac{C(n)}{|k|} \frac{2\delta}{2\delta-1} \|(\Delta+k^2)u\|^2_{L_\delta}$

ii) $\|u\|_{B^*} \le \frac{C(n)}{|k|} \|(\Delta+k^2)u\|_B$

iii) If supp $f \subset D$, D bounded, then for $u = R_0(\lambda+i0)f$

$$\|u\|_{L^2(D)} \le C(n)\frac{d}{|k|}\|f\|_{L^2(D)},$$

where $d = d(D)$.

<u>Proof</u> We prove iii): according our theorem

$$u = \sum u_i \quad \Rightarrow$$

$$\|u\|_{L^2(D)} \le \sum \|u_i\|_{L^2(D)} \le$$

$$\overset{\text{Lemma } 18.6}{\le} \sum \sqrt{d}\, \|u_i\|_{\theta^\infty(D)} \overset{\text{Theorem 17}}{\le} \frac{C(n)}{|k|} \sum \sqrt{d}\, \|f\|_{\theta^1(D)}$$

$$\overset{\text{Lemma } 18.6}{\le} \frac{C(n)}{|k|} \sum d\, \|f\|_{L^2(D)} = \frac{(n+1)\,C(n)\,d}{|k|} \|f\|_{L^2(D)}$$

$\square$

Recall $\mathcal{J}_n = \{\Theta_0, \dots, \Theta_n\}$. Define

$$\mathcal{J}^1 = \bigcap_{j=0}^{\infty} \Theta_j^1$$

$$\|f\|_{\mathcal{J}^1} = \max \|f\|_{\Theta_j^1}$$

$$\mathcal{J}^\infty = \sum_{j=0}^{n} \Theta_j^\infty$$

$$\|u\|_{\mathcal{J}^\infty} = \inf_{u=\sum u_j} \sum \|u_j\|_{\Theta_j^\infty}$$

Note that Theorem M a) implies

$$\|u_j\|_{\Theta_j^\infty} \le \frac{C(n)}{|k|} \|f\|_{\mathcal{J}^1}$$

Summing up gives

(M) $$\|u\|_{\mathcal{J}^\infty} \le \frac{C(n)(n+1)}{|k|} \|f\|_{\mathcal{J}^1} = \frac{C(n)}{|k|} \|f\|_{\mathcal{J}^1}$$

<u>Proof of Theorem M:</u>

Let $\Theta \in S^{n-1}$ and

$$x = t\Theta + y \quad ; \quad \Theta \cdot y = 0$$

$$\xi = \tau\Theta + \eta \quad ; \quad \Theta \cdot \eta = 0$$

We write

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(x)\,dx \quad (\Rightarrow)$$

$$\hat{f}(\tau, \eta) = \int_{\mathbb{R}^n} e^{it\tau} e^{iy\cdot\eta} f(t,y)\,dy\,dt$$

We denote partial F-transform on $\theta^\perp$ hyperplane, by

$$\tilde{f}(t,\eta) = \mathcal{F}_\theta f = \int_{x\cdot\theta = t} e^{iy\cdot\eta} f(t,y)\,dy,$$

Now

$$(\Delta + k^2) u = f$$

$$\Rightarrow \quad \frac{d^2 \tilde{u}}{dt^2} + (k^2 - \eta^2)\tilde{u} = \tilde{f} \qquad (1)$$

For $f \in L^2$, $k \in \mathbb{C}^+$ $\exists!$ $u \in H^2(\mathbb{R}^n)$ and

$$(*) \quad \tilde{u}(t,\eta) = \int_{\mathbb{R}} \frac{e^{i\sqrt{k^2-\eta^2}\,|t-s|}}{2i\sqrt{k^2-\eta^2}}\,\tilde{f}(s,\eta)\,ds$$

Proof

$$\frac{d^2}{dt^2} e^{i\sqrt{k^2-\eta^2}\,|t-s|} = \frac{d}{dt}\, i\sqrt{k^2-\eta^2}\, e^{i\sqrt{k^2-\eta^2}\,|t-s|}\,\mathrm{sgn}(t-s)$$

$$= -(k^2-\eta^2) e^{i\sqrt{k^2-\eta^2}\,|t-s|} \underbrace{(\mathrm{sgn}(t-s))^2}_{1} + \underbrace{e^{i\sqrt{k^2-\eta^2}\,|t-s|}}_{i\sqrt{k^2-\eta^2}}\, 2\delta(t-s)$$

Thus

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$$\frac{d^2}{dt^2}\tilde{u}(t,\eta) = -(k^2-\eta^2)\int \frac{e^{i\sqrt{k^2-\eta^2}|t-s|}}{2i\sqrt{k^2-\eta^2}}\tilde{f}(s,\eta)\,ds$$

$$+ \quad \tilde{f}(t,\eta) = -(k^2-\eta^2)\,\tilde{u}(t,\eta) + \tilde{f}(t,\eta)$$

$\blacksquare$

<u>Remark</u> For $k \in \mathbb{C}^+$ the solution of

$$(\Delta + k^2)u = f \in L^2$$

is unique in L^2. Indeed if $u \in L^2$ satisfies

$$(\Delta + k^2)u = 0 \Rightarrow u \in H^2$$

Since $k \in \mathbb{C}^+$ we get

$$\underbrace{(-\xi^2 + k^2)}_{\neq 0 \ \forall \xi}\hat{u}(\xi) = 0 \Rightarrow u = 0$$

For $k \in \overline{\mathbb{C}_+} \Rightarrow$ problems

Define

$$W_\theta^\varepsilon = \{\xi = t\theta + \eta \mid |\eta^2 - 1| \le \varepsilon^2\}$$

$$= \{\xi \in \mathbb{R}^n \mid |1 - \xi^2 + (\theta\cdot\xi)^2| \le \varepsilon^2\}$$

Note

$$1 - \xi^2 + (\theta\cdot\xi)^2 = 1 - (t\theta + \eta)^2 + t^2$$

$$= 1 - t^2 - \eta^2 + t^2 = 1 - \eta^2$$

For $\alpha > 0$ we set

$$\alpha W_\theta^\varepsilon = \{ \xi \mid |\alpha - \eta^2| \le \varepsilon^2 \alpha^2 \}$$

<u>Proposition</u> Assume $\{\theta_1, \ldots, \theta_n\}$ is an orthonormal basis, $\theta_0 = \frac{\theta_1 + \theta_2}{\sqrt{2}}$ and $\varepsilon \le \sqrt{\frac{n-1}{4n(n+1)}}$. Then for $f \in C_0^\infty$ $\exists$ decomposition

$$f = \sum_{j=0}^{n} f_j \qquad \text{with}$$

$$\|f_j\|_{\theta_j^1} \le c(n) \|f\|_{\theta_j^1}$$

and

$$\operatorname{supp} \hat{f}_j \cap \alpha W_{\theta_j}^\varepsilon = \emptyset .$$

<u>Proof of Theorem M</u> Assume

$$f = \sum f_j \qquad \text{as in prop.}$$

and denote

$$u_j = R(k^2) f_j \iff (\Delta + k^2) u_j = f_j$$

Apply partial F. transform w.r.t. θ_j

$$\hat{u}_j(t, \eta) = \int \frac{e^{i\sqrt{k^2 - \eta^2}\,|t-s|}}{2i\sqrt{k^2 - \eta^2}} \hat{f}_j(s, \eta)\, ds$$

Because $\operatorname{Im}\sqrt{k^2-\eta^2} \geq 0$ we can estimate as follows

$$\int |\tilde{u}_i|^2 \leq$$

$$\frac{1}{4}\int_{\mathbb{R}^{n-1}} d\eta \left[\frac{1}{\sqrt{|k^2-\eta^2|}} \int_{\mathbb{R}} |\tilde{f}_i(\xi,\eta)|\,ds \int_{\mathbb{R}} |\tilde{f}(\nu,\eta)|\,d\nu\right]$$

Next we use the triangle inequality

$$\frac{1}{|k^2-\eta^2|} \leq \frac{1}{\left||k|^2-\eta^2\right|}$$

By

$$\operatorname{supp} \hat{f}_j \cap \alpha W^{\varepsilon}_{\theta_j} = \emptyset$$

we get for $\tilde{f}_j$

$$\operatorname{supp} \tilde{f}_j \cap \{(x,\eta) \mid \eta \in \alpha W^{\varepsilon}_{\theta_j}\} = \emptyset$$

Thus in the support of $\tilde{f}_j$ we have by taking $\alpha = |k|$ that

$$\left||k|^2-|\eta|^2\right| \geq \varepsilon^2 |k|^2$$

and

$$\frac{1}{|k^2-\eta^2|} \leq \frac{1}{\varepsilon^2|k|^2}$$

This gives us

$$\int |U_i|^2 \, dy \le \frac{1}{\varepsilon |k|^2} \int_{R^{n-1}} d\eta \int_R |\tilde{f}_i(s,\eta)| \, ds \int_R |\tilde{f}_i(r,\eta)| \, dr$$

$$\le \frac{1}{\varepsilon |k|^2} \int_{R^2} \|\tilde{f}_i(s,\cdot)\|_{L^2} \, \|\tilde{f}_i(r,\cdot)\|_{L^2} \, ds \, dr$$

$$= \frac{1}{\varepsilon |k|^2} \left(\int_{R^2} \|f_i(s,\cdot)\|_{L^2} \, ds \right)^2 = \frac{1}{\varepsilon |k|^2} \|f_i\|^2_{\theta_i^1}$$

proving the first part of theorem 11.

To estimate ∇u we write

(1) $$v \cdot \nabla u = \sum_{i=1}^{n} v \cdot \theta_i \; \theta_i \cdot \nabla u$$

Now after applying $\mathcal{F}_{\theta_i}$ $(x = \theta_i t + y$ etc$)$

$$\tilde{u}(t,\eta) = \int \frac{e^{i\sqrt{k^2-\eta^2}\,|t-s|}}{2i\sqrt{k^2-\eta^2}} \tilde{f}(s,\eta) \, ds$$

$$\Rightarrow \frac{d\,\tilde{u}(t,\eta)}{dt} = \int e^{i\sqrt{k^2-\eta^2}\,|t-s|} \, \mathrm{sgn}(t-s) \, \tilde{f}(s,\eta) \, \tfrac{1}{2}$$

$\Rightarrow$
(2) $$\left\| \frac{d\tilde{u}(t,\eta)}{dt} \right\|_{\theta_i^\infty} \le \|f\|_{\theta_i^1}$$

But

$$\|\theta_i \cdot \nabla u\|_{\theta_i^\infty} = \sup_t \left\| \frac{d}{dt} u(t,x) \right\|_{L^2_x} = \left\| \frac{d}{dt} \tilde{u} \right\|_{\theta_i^\infty}$$

which together with (1) and (2) proves

$$\nabla u = \sum_{i=1}^{n} w_i \,, \quad \text{with}$$

$$\|w_i\|_{\Theta_i^\infty} \leq \|f\|_{\Theta_i^1}$$

Finally

$$\Delta u = f - k^2 u \Rightarrow$$

$$\|\Delta u\|_{\Theta_i^\infty} \leq \|f\|_{\Theta_i^\infty} + |k|^2 \|U\|_{\Theta_i^\infty}$$

$$\leq \left(\|f\|_{\Theta_i^\infty} + C(n) |k| \|f\|_{\Theta_i^1} \right)$$

$\blacksquare$

So we are left to prove Corollary 18.8.

We define

$$\sigma_\theta(\xi) = \xi^2 - (\theta \cdot \xi)^2 \qquad (= \eta^2)$$

and recall

$$W_\theta^\varepsilon = \{ \xi \,|\, |\sigma_\theta(\xi) - 1| < \varepsilon^2 \}$$

We need to show first

$$\bigcup_{i=0}^{n} \mathbb{R}^n \setminus W_{\theta_i}^\varepsilon = \mathbb{R}^n$$

<u>18.9 Lemma</u> For $\varepsilon < \frac{1}{\sqrt{4n+5}}$

$$\bigcap_{k=0}^{n} W_{\theta_k}^\varepsilon = \emptyset$$

(11

<u>Proof</u> For $\xi \in \mathbb{R}^n$ let $\xi_k = \xi \cdot \theta_k \Rightarrow$

$$\xi = \sum_{k=1}^{n} \xi_k \theta_k$$

Let $\tau = \sum_{k=1}^{n} \tau_k \theta_k$, $\tau_k = \xi_k^2$, and

$$\mathbb{1} = \sum_{k=1}^{n} \theta_k .$$

Now

$$\xi \in \bigcap_{k=1}^{n} W_\theta^0 \iff \sigma_{\theta_i} = \xi^2 - (\theta_i \cdot \xi)^2 = 1$$

$$\forall i = 1, \ldots, n$$

$$\iff \sum_{k \neq i} (\theta_k \cdot \xi)^2 = \sum_{k \neq i} \xi_k^2 = 1 \; ; \; \forall i = 1, \ldots, n .$$

$\iff M\tau = \mathbb{1}$, where

$$M = \begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \\ 1 & 1 & 0 & \\ \vdots & & & \\ 1 & 1 & \ldots 1 & 0 \end{pmatrix} = \mathbb{1} \otimes \mathbb{1} - I$$

Is M invertible ? Yes

$$M^{-1} = \frac{1}{(n-1)} \mathbb{1} \otimes \mathbb{1} - I = \frac{1}{n-1} \begin{pmatrix} 2-n & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & & \\ 1 & & & 2-n \end{pmatrix}$$

Check

$$\left(\frac{1}{n-1} \mathbb{1} \otimes \mathbb{1} - I \right) \left(\mathbb{1} \otimes \mathbb{1} - I \right) =$$

$$= \left(\frac{n}{n-1} - 1 - \frac{1}{n-1}\right) + I = I$$

Moreover, if we equip $\mathbb{R}^n$ with ℓ^∞ norm we have

$$\| M^{-1} \|_{\ell^\infty \to \ell^\infty} = 2 - \frac{1}{n-1}$$

The unique solution to

$$M \tau = 1$$

is

$$\tau = M^{-1} 1 = \frac{n}{n-1} 1 - 1 = \frac{n-(n-1)}{n-1} 1$$

$$\Rightarrow \quad \tau = \frac{1}{n-1} 1$$

This means that

$$\xi \in \bigcap_{k=1}^{n} W_{\theta_k}^{\varepsilon} \Rightarrow$$

$$\| \tau - \frac{1}{n-1} 1 \|_{\ell^\infty} \le \left(2 - \frac{1}{n-1}\right) \varepsilon^2$$

In particular for each $k = 1, \dots, n$

$$(1) \qquad |\xi_k| \ge \sqrt{\frac{1}{n-1} - \left(2 - \frac{1}{n-1}\right) \varepsilon^2}$$

Moreover,

$$\sigma_{\theta_0} = \xi^2 - (\theta_0 \cdot \xi)^2 = \xi^2 - \left(\frac{\theta_1 + \theta_2}{\sqrt{2}} \cdot \xi\right)^2$$

(13)

$$= \xi^2 - \frac{1}{2}\left(\theta_1 \cdot \xi + \theta_2 \cdot \xi\right)^2 \qquad \Rightarrow$$

$$\sigma_{\theta_0} - 1 = \frac{1}{2}(\sigma_{\theta_1} - 1) + \frac{1}{2}(\sigma_{\theta_2} - 1) - \xi_1 \xi_2$$

This implies for $\xi \in \bigcap_{i=1}^{n} W_{\theta_i}^{\varepsilon}$

$$|\sigma_{\theta_0} - 1| \geq |\xi_1 \xi_2| - \varepsilon^2$$

$$\overset{(1)}{\geq} \frac{1}{n-1} - \left(\left(2 - \frac{1}{n-1}\right) + 1\right)\varepsilon^2$$

$$= \frac{1}{n-1} - \left(3 - \frac{1}{n-1}\right)\varepsilon^2$$

So $\bigcap_{i=0}^{n} W_{\theta_i}^{\varepsilon} = \emptyset$ if

$$\varepsilon^2 \leq \frac{1}{n-1} - \left(3 - \frac{1}{n-1}\right)\varepsilon^2$$

$$\Leftrightarrow \quad \varepsilon \leq \frac{1}{\sqrt{4n-5}} \qquad \square$$

Let $\varphi \in C_0^\infty(\mathbb{R})$ satisfy

$$\varphi(s) = \begin{cases} 1, & \text{for } |1-s| < \varepsilon/2 \\ 0 & \text{for } |1-s| > \varepsilon \end{cases}$$

and $\theta \in S^{n-1}$. Denote $\xi = \tau\theta \cdot \eta$

We define

$$\varphi_\theta(\xi) = \varphi_\theta(\tau,\eta) = \varphi(|\eta|^2) = \varphi(\delta_\theta(\xi))$$

Lemma i) $\operatorname{supp}\varphi_\theta \subset W_\theta^\varepsilon$

$$\operatorname{supp}(1-\varphi_\theta) \cap W_\theta^{\varepsilon/2} = \emptyset$$

ii) $\widehat{\varphi_\theta}$ is bounded measure with

$$|||\widehat{\varphi_\theta}|||_1 := \int_{R^n} |\widehat{\varphi_\theta}|$$

iii) $|||\mathcal{F}\varphi_\theta(\cdot/\alpha)|||_1$ is independent of α ($\alpha \in [\ldots]$)

iv) For $\varepsilon \le \dfrac{1}{\sqrt{4n-5}}$,

$$\sum_{m=0}^{n}\left[(1-\varphi_m(\xi))\prod_{j<m}\varphi_j(\xi)\right] = 1$$

Proof i) follows directly from the definitions

ii) $\varphi \in S(R) \Rightarrow \psi(\eta) = \varphi(|\eta|^2) \in S(\theta^\perp)$
$\Rightarrow \hat\psi(y) \in S$

Now

$$\varphi_\theta(\tau,\eta) = \psi(\eta) \otimes 1$$

$$\Rightarrow \widehat{\varphi_\theta} = \hat\psi(y) \otimes \delta(t)$$

$\Rightarrow$ $\hat{\varphi}_\theta$ is a measure supported on $\{x \cdot \theta = \alpha\}$ and

$$\|\hat{\varphi}_\theta\|_1 = \int_{\mathbb{R}^{n-1}} |\hat{\varphi}(y)|\, dy$$

iii)

Moreover

$$\widehat{\varphi_\theta(\cdot/\alpha)} = \widehat{\varphi(y/\alpha)} \otimes \delta(t)$$

$$= \alpha^{n-1} \hat{\varphi}(\alpha y) \otimes \delta(t)$$

which proves iii).

iv)

$$\sum_{m=0}^{n} \left[(1-\varphi_m(\xi)) \prod_{j<m} \varphi_j(\xi) \right] = 1 - \prod_{j=0}^{n} \varphi_j(\xi)$$

Note that $\prod_{j=0}^{n} \varphi_j(\xi) \neq 0$ only if $\xi \in \bigcap_{k=0}^{n} W_\theta^\varepsilon = \emptyset$.

$\blacksquare$

Now we can define

$$\hat{f}_m(\xi) = \left[(1-\varphi_m(\xi/\alpha)) \prod_{j<m} \varphi_j(\xi/\alpha) \right] \hat{f}$$

$\Rightarrow$ $\sum f_m = f$ and $f_m \cap \alpha W_{\theta_m}^\varepsilon = \emptyset$

Finally

$$\hat{f}_m = \left(\prod_{i=0}^{m} g_k\right)\hat{f}$$

where $\hat{g}_k =: \mu_k$ is bounded measure.
we have

$$f_m = f * \mu_0 * \mu_1 \dots * \mu_m =: f * \mu$$

where μ is a bounded measure, the estimate

$$\| f_m \|_{\Theta_m^1} \leq C(n) \| f \|_{\Theta_m^1}$$

follows from

Lemma $\| f * \mu \|_{\Theta^1} \leq \| f \|_{\Theta^1} \, ||| \mu |||_1$

Proof We may assume $\Theta = e_1$ and $|||\mu|||_1 = 1$

$$| f * \mu (x_1, x') | = \left| \int f(x_1 - y_1, x' - y') \mu(y_1, y') \right|$$

$$\leq \int | f(x_1 - y_1, x' - y') | \mu^{1/2}(y_1, y') \mu^{1/2}(y_1, y') |$$

$$\leq \left(\int | f(x_1 - y_1, x' - \cancel{y'}) |^2 \mu(y_1, y') \, dy \, dy_1 \right)^{1/2}$$

$$\| f * \mu \|_{\Theta^1} = \int \left(\iiint | f(x_1 - y_1, x' | \mu(y_1, y') \, dy' dx' dy_1 \right)^{1/2} dx_1$$

Minkowskin epäyhtälö

$$\left\| \int f(\cdot, y) \, dy \right\| \leq \int \| f(\cdot, y) \| \, dy \qquad \square$$

<u>Proof</u> We assume $\Theta = e_1$

(1) $$f * \mu(x_1, x') = \int f(x_1 - y_1, x' - y')\, \mu(y_1, y')\, dy_1 dy'$$

Minkowsky inequality. $1 \leq p \leq \infty$

$$\left(\left| \int |F(x,y)|\, d\mu_1(x) \right|^p d\mu_2(y) \right)^{1/p} \leq$$

$$\leq \int \left(\int |F(x,y)|^p\, d\mu_2(y) \right)^{1/p} d\mu_1(x)$$

in other words

$$\left\| \int F(\cdot, y)\, d\mu_2(y) \right\|_{L^p(d\mu_2)}$$

$$\leq \int \| F(\cdot, y) \|_{L^p(d\mu_2)}\, d\mu_2(y)$$

Applying Minkowsky to (1) yields

$$\| f * \mu(x_1, \cdot) \|_{L^2(\mathbb{R}^{n-1})} \leq$$

$$\leq \int \| f(x_1 - y_1, \cdot) \|_{L^2(\mathbb{R}^{n-1})}\, \mu(y_1, y')\, dy_1 dy'$$

$$\Rightarrow \| f * \mu \|_{\Theta^1} = \int \| f * \mu(x_1, \cdot) \|_{L^2(\mathbb{R}^{n-1})}\, dx_1$$

$$\leq \| f \|_{\Theta^1} \int \mu(y_1, y')\, dy_1 dy'$$

$\boxed{4}$

19 Limit independent criteria related to Born series and TE's

How good is Born approximation?

Assume supp m is compact

Extended scattering theory

① The incident wave u^0 is defined only in D, supp $m \subset D$, D bounded domain. The scattered wave u^s is defined in $\mathbb{R}^n$ and is outgoing $\Rightarrow$ The total wave

$$u = u^0 + u^s$$

is defined only in D.

Compare with standard scattering theory

② All waves are defined in $\mathbb{R}^n$.

In ① the solution space A can be taken $L^2(D)$; $A = L^2(D)$

In ② usually $A = B^*$ or $A = L^2_{-\delta}$.

We will use in ②

$$A = J^\infty := \sum_{j=0}^{n} \oplus_j^\infty$$

Recall

$$\|u\|_{J^\infty} = \inf_{u = \sum u_j} \|u_j\|_{\Theta_j^\infty}$$

and

$$d = d(D) = \sup_\ell |\ell \cap D|$$

where the sup is over all lines in $\mathbb{R}^n$

<u>19.1 Theorem</u> Let supp $m \subset D$ and suppose

$$u = u^m \in A \quad \text{solves}$$

$$(\Delta + k^2(1+m))u = 0$$

Then u has a unique decomposition

$$u = u^0 + u^s$$

where $u^0 \in A$, $(\Delta + k^2)u^0 = 0$ and $u^s \in J^\infty$.

Moreover if

(1) $$\|m\|_{L^\infty(D)} \leq \frac{1}{C(n)|k|\, d},$$

then the Born series

$$u^s = \sum_{j=1}^{\infty} \left[k^2 R(k^2) m\right]^j u^0$$

converges,

$$\|u^s\|_A \le \frac{\alpha}{1-\alpha} \|u^0\|_A \quad ; \quad \alpha = C(n)|k| d \|m\|_{L^\infty(D)}$$

and the Born approximation

$$u^s_B = k^2 R(k^2) m u^0$$

satisfies

$$\|u^s - u^s_B\|_A \le \frac{\alpha}{1-\alpha} \|u^s_B\|_A$$

Remark Note $|k| d$ is unitless

hence (1), (2) and (3) are independent of unit chosen. Note $|k|$ is the wavenumber

The unit of k is $[k] = \frac{1}{m}$ = one over meter

kd measures the size of D measured in wave lengths.

We employ

19.2 Lemma Assume $m \in L^\infty(D)$. If $A = L^2(D)$ or $A = J^\infty$, then

$$\|k^2 R(k^2) m v\|_A \le C(n)|k| d \|m\|_{L^\infty} \|v\|_A$$

Proof The corollary of Theorem M says

$$\|u\|_{L^2(D)} \le C(n) \frac{d}{|k|} \|f\|_{L^2(D)} \text{ for}$$

$$u = R(k^2) f$$

Thus

$$\| k^2 R(k^2)\, m v \|_{L^2(D)} \le |k|\, d\, \| m v \|_{L^2(D)}$$

$$\le |k|\, d\, \| m \|_{L^\infty(D)} \| v \|_{L^2(D)}$$

Another corollary of Theorem M says

$$\| u \|_{\mathcal{J}^\infty} \le \frac{C(n)}{|k|} \| f \|_{\mathcal{J}^1}, \quad u = R(k^2) f$$

Hence $\exists\, j \in \{1, \dots, n\}$

1) $$\| k^2 R(k^2)\, m v \|_{\mathcal{J}^\infty} \le C(n) |k|\, \| m v \|_{\mathcal{J}^1}$$

$$= C(n) \| m v \|_{\Theta_j^1}$$

We apply

2) $$\frac{1}{\sqrt{d(D,\Theta)}} \| f \|_{\Theta^1} \le \| f \|_{L^2(D)} \le \sqrt{d(D,\Theta)}\, \| f \|_{\Theta^\infty}$$

to obtain

$$\| m v \|_{\Theta_j^1} \le \sqrt{d}\, \| m v \|_{L^2(D)} \le \sqrt{d}\, \| m \|_{L^\infty} \| v \|_{L^2(D)}$$

$$\le \sqrt{d}\, \| m \|_{L^\infty} \inf_{v = \sum_{i=0}^{n} v_i} \sum \| v_i \|_{L^2(D)}$$

$$\overset{(2)}{\le} d\, \| m \|_{L^\infty} \| v \|_{\mathcal{J}^\infty}$$

This together with (1) proves the claim □

Proof of Theorem 18.1:

$$(\Delta+k^2)u = -k^2 m u \quad , \text{ define}$$

$$u^s = R(k^2)[+k^2 m u] \quad \text{and}$$

$$u^0 = u - u^s$$

Since $u \in A$ we have $mu \in \mathcal{J}^1 \Rightarrow$
$u^s \in \mathcal{J}^\infty \subset A$. Clearly $(\Delta+k^2)u^0 = 0$.
Moreover

$$u^s = +k^2 R(k^2)\, m\,(u^0 + u^s)$$

This means that

(1) $$u^s = \sum_{j=1}^{\infty} (k^2 R(k^2) m)^j \, u^0$$

if (1) converges. Check

(2) $$\| k^2 R(k^2) m \|_{A \to A} < 1 \quad ?$$

19.2 Lemma $\Rightarrow$

$$\| k^2 R(k^2) m \|_{A \to A} \le C(n) \|m\|_{L^\infty} |k| \,\ell =: \alpha$$

$$< 1$$

Thus (2) ok. Moreover

$$\|u^s\|_A \le \frac{\alpha}{1-\alpha} \|u^0\|_A \quad \text{and}$$

$$\|u^s - u^s_B\|_A \le \sum_{j=2}^{\infty} \|(k^2 R(k^2) m)^{j-1} u^s_B\|_A$$

$$\le \frac{\alpha}{1-\alpha} \| u^s_B \|_A$$

$\boxtimes$

Remark 1 Theorem 19.1 tells that Born is OK if d is small i.e. if the support of m has several components than the sum of the diameters of each component must be small.

Remark 2 The norm Θ^1, Θ^∞ and hence J^1 and J^∞ are translation invariant as L^2_δ, B, B^* are not $\Rightarrow$ Theorem 19.1 is possible.

As a final step we prove that small scatterers don't have small TE's.

19.3 Theorem If $k>0$ is a TE for $m \in L^\infty(D)$, then

$$1 \le C(n)\, d\, k \, \| m \|_{L^\infty}.$$

Proof Let v^0 be any incident wave

(1) $$\int_D v^0 (\Delta + k^2) u^s = - \int_D k^2 m u v^0$$

If k is TE and

(2) $$u = u^0 + u^s$$

where u^s vanishes outside D we get l.h.s of (1) vanishes. Thus

(3) $$\int_D m u v^0 = 0$$

Next multiply (2) with $m\bar{u}$ =)

$$\int m|u|^2 = \underbrace{\int m \bar{u} u^0}_{\overset{(3)}{=} 0} + \int m \bar{u} u^s$$

Hence

$$\int m|u|^2 = \int m \bar{u} u^s \leq \|mu\|_{L^2(D)} \|u^s\|_{L^2(D)}$$

$$\leq \|mu\|_{L^2(D)} \frac{C(n) d}{k} \|k^2 m u\|_{L^2(D)}$$

$$= C(n) d k \|mu\|^2_{L^2(D)} \leq C(n) d k \|m\|_{L^\infty} \int m|u|^2$$

=)

$$1 \leq C(n) d k \|m\|_{L^\infty}$$

$\boxed{/}$

END OF EVERYTHING