Proof We prove (ii) first:

$$(\Delta + k^2)u^0 = 0 \iff$$

$$(\Delta + k^2(1+m))u^0 = k^2 m u^0 =: -g$$

By Theorem 15.3, $\exists\, u^+ \in B^*$ s.t.

$$(\Delta + k^2(1+m))u^+ = g$$

Set $u^m = u^0 + u^+$, then

$$(\Delta + k^2(1+m))u^m = 0$$

which gives (ii) i.e. proves the existence of such a decomposition. If

$$u_0 = w^m - w^+ \quad \text{is another such decomp.,}$$

then

$$(\Delta + k^2(1+m))w^+ = -k^2 m u^0$$

Thus

$$(\Delta + k^2(1+m))(u^+ - w^+) = 0$$

and both outgoing $\Rightarrow$ $u^+ = w^+$.

The proof of (i) is similar and is left for exercise. $\square$

Remark It follows from Hö, Theorem 14.6.8 that

$$\widehat{u_0} = v_+ \delta(P_0 - \lambda) = v_+ \delta(\xi^2 - k^2)$$

We study

$$(\Delta + k^2(1+m))\, w^+ = g$$

$$\left(\begin{aligned} &\Leftrightarrow \quad w^+ + k^2 R_0(\lambda+i0)\, m w^+ = R_0(\lambda+i0)\, g \\ &\Leftrightarrow \quad w^+ = k^2 R_0(\lambda+i0)\, [-m w^+ + g] \end{aligned} \right)$$

Denote

$$G_m : g \mapsto w^+$$

$$G_0 = G_m \quad \text{for} \quad m \equiv 0.$$

<u>Def</u> 1. $B^0 = \{ u \in B^* \mid (\Delta + k^2) u = 0 \}$

2. $B^m = \{ u \in B^* \mid (\Delta + k^2(1+m)) u = 0 \}$

3. $B^+ = \{ u \in B^* \mid u \text{ is outgoing} \}$

<u>15.5 Theorem</u> (i) $u_0 \mapsto u^+$ maps B^0 into B^+

(ii) $R_0(\lambda+i0) : B \to B^+$ is an isomorphism

(iii) $u^0 \mapsto u^m$ is an isomorphism from B^0 onto B^m.

<u>Proof</u> (i) Theorem 15.4

(ii) Injective: $R_0(\lambda+i0) u = 0$, $u \in B \Rightarrow$

$$u = (\Delta + k^2) R_0(\lambda+i0) u = 0$$

Surjective: If $v^+ \in B^+ \Rightarrow$

$v^+ = R_0(\lambda+i0) f$, $f \in B$ by def.

(iii) Denote

$$\mathcal{S} : u^0 \to u^{in} , \quad \mathcal{S} : B^0 \to B^{in}$$

is bijective by Theorem 15.3 and

$$\| u^0 \|_{B^*} = \| u^{in} - u^+ \|_{B^*} \le \| u^{in} \|_{B^*} + \| u^+ \|_{B^*}$$

Moreover

$$\| u^+ \|_{B^*} = \| R_0(\lambda + i0) V u^{in} \|_{B^*}$$

$$\le C \| V u^{in} \|_{B} \le C_1 \| u^{in} \|_{B^*}$$

Thus

$$\| u^0 \|_{B^*} \le (1 + C_1) \| u^{in} \|_{B^*}$$

$$\Leftrightarrow \quad \| u^0 \|_{B^*} \le (1 + C_1) \| \mathcal{S} u \|_{B^*}$$

proving $\mathcal{S} : B^0 \to B^{in}$ is an isomorphism

$\square$

The map $\mathcal{S} : u^0 \mapsto u^{in}$ is called the <u>scattering map</u>. The relation to <u>scattering operator</u> S and <u>scattering matrix</u> Σ will be studied next.

We start with S:

$$S = W_-^{-1} W_+ = W_-^* W_+ \quad (\text{unitary in } L^2)$$

where the wave operators

$$W_\pm = \lim_{t \to \pm\infty} e^{itH} e^{-itH_0} \quad (\text{isometric})$$

$$\Rightarrow \quad H W_\pm = W_\pm H_0 \quad \Rightarrow \quad S H_0 = H_0 S$$

Hence S commutes with H_0. In Fourier side this means

$$\Sigma = F S F^{-1} \quad (\text{scattering matrix})$$

commutes with $P_0(\xi)$ and hence Σ is well defined on $L^2(M_\lambda)$ and is <u>unitary</u> w.r.t. measure $\frac{dS}{|P_0'|}$.

$$\Sigma_\lambda : L^2(M_\lambda) \to L^2(M_\lambda)$$

The formula

$$(F S F^{-1}) f|_{M_\lambda} = \Sigma_\lambda f|_{M_\lambda}$$

for $\lambda \in \operatorname{Im} P_0 \setminus Z(P_0)$ $(= (0,\infty),\ P_0 = -\Delta)$

defines S when Σ_λ's are given.

Finally, if $(P_0(D) - \lambda + V) u = 0$, $u \in B_0^*$, then

$$u = u_\pm - R_0(\lambda \pm i0) V u$$

and Σ_λ is also defined through

$$\Sigma_\lambda : v_+ \to v_- ,$$

where

$$\hat{u}_\pm = v_\pm \delta(P_0 - \lambda) = v_\pm |P_0'| \, ds$$

This is our starting point to understand the connection between S and Σ_λ. We need to discuss asymptotics

15.6 Theorem i) For any $u^0 \in B^0$, there exists $\mu^0 \in L^2(S^{n-1})$ s.t.

$$u^0(x) \sim \mu^0(\theta) \frac{e^{ihr}}{(ihr)^{\frac{n-1}{2}}} + \mu_0(-\theta) \frac{e^{-ihr}}{(-ihr)^{\frac{n-1}{2}}}$$

ii) For any $u^+ \in B^+$,

$\exists \ \mu^+ \in L^2(S^{n-1})$ s.t.

$$u^+ \sim \mu^+(\theta) \frac{e^{ihr}}{(ihr)^{\frac{n-1}{2}}}$$

iii) For any $u^m \in B^m$, $\exists \ \mu^m \in L^2$ and $\beta \in L^2$ s.t.

$$u^m(x) = (\mu^m(\theta) + \beta(\theta)) \frac{e^{ihr}}{(ihr)^{\frac{n-1}{2}}} + \mu^m(-\theta) \frac{e^{-ihr}}{(ihr)^{\frac{n-1}{2}}}$$

We will apply Theorem 7.7.14 of [Hö] which we state without proof. It is a version of the method of stationary phase.

<u>7.7.14</u> Let Σ be a C^∞ hypersurface in $\mathbb{R}^n$, with total curvature $K \neq 0$. Then, if $v = a\,dS \in C_0^\infty$

(7.7.25) $$\Big| \tau^{\frac{n-1}{2}} \hat{v}(\tau\xi) - \sum a(x)|K(x)|^{-1/2} (2\pi)^{\frac{n-1}{2}} e^{-i\langle x, \tau\xi\rangle - \pi i \sigma/4} \Big|$$

$$\leq \frac{C}{\tau}, \quad |\xi| = 1, \quad \tau > 0.$$

Here the sum is over such $x \in \operatorname{supp} v$ that ξ is normal to Σ at x and

$$\sigma = \sigma_+ - \sigma_-$$

where $\sigma_\pm$ is the number of centers of curvature in the direction $\pm\xi$.

We will formulate (7.7.25), for convenience, in the case when $\Sigma = S_k = \{\xi \mid |\xi| = k\}$ (we change the roles of ξ and x).

First we note that here

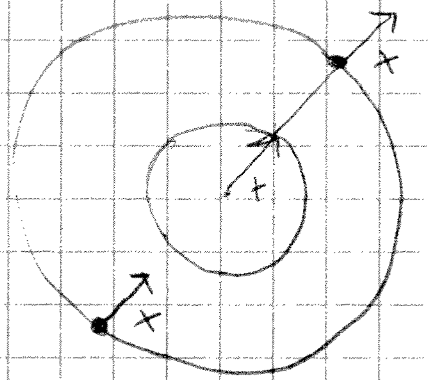
$$\sigma = 0 \quad \text{and}$$

$$K(\xi) = \frac{1}{k^{n-1}} \quad \text{is constant.}$$

We denote $u = F^{-1} v$,
$\hat{u} = v = a\,dS$

and then
$$u(x) = \hat{v}(-x).$$

When $x = -\theta \in S^{n-1}$ is fixed there are exactly two points in $\Sigma = S_k$ where x is normal to Σ, namely $\xi = kx = -k\theta$ and $\xi = -kx = k\theta$.



Thus (7.7.25) rewrites in this case
$$\hat{u} = a\,dS, \quad \theta \in S^{n-1}$$
in the form

(7.7.25)' $$\left| \tau^{\frac{n-1}{2}} u(\tau\theta) - (2\pi k)^{\frac{n-1}{2}} \left[a(k\theta) e^{-i\tau\theta\cdot\xi} + a(-k\theta) e^{i\tau\theta\cdot\xi} \right] \right| \le \frac{C}{\tau}.$$

Here C is independent of θ, τ and k.