Now if supp $V \subset D$ we have

$$u^0 \in B^0_D \,, \quad u^m \in B^m_D$$

and

$$u^0 - u^m = u^s$$

vanishes in the neighborhood of ∂D. Then $u^0 - u^m \in \overset{\circ}{H}{}^2(D)$ and the interior transmission problem

$$(\Delta + k^2) v = 0 \quad \text{in } D$$
$$(\Delta + k^2(1+m)) w = 0 \quad \text{in } D$$
$$v = w \,, \quad \frac{\partial}{\partial \nu} v = \frac{\partial}{\partial \nu} w \quad \text{on } \partial D$$

has a non-trivial solution $v = u^0$ and $w = u^m$.

16.3 Theorem The following are equivalent

1. λ is a D-TE
2. $\exists \; 0 \neq U^m \in B^m_D$

$$R_0(\lambda + i0)\, m\, U^m \in H^2_0(D) \qquad (16.1)$$

3. $\exists \; 0 \neq U^m \in B^m_D$ and some $v \in H^2_0$ satisfying

$$(\Delta + k^2) v = -k^2 m U^m$$

4. $\exists\ 0 \neq u^0 \in B^0_D$ s.t. the unique outgoing u^s satisfying

$$(\Delta + k^2(1+m))\,u^s = -k^2 m u^0$$

(denote $u^s = k^2 R(\lambda + i0)\, m u^0$)

is in $H^2_0(D)$

5. $\exists\ 0 \neq u^0 \in B^0_D$ and some $v \in H^2_0(D)$ satisfying

$$(\Delta + k^2(1+m))\,v = -k^2 m u^0$$

Proof 2) $\Rightarrow$ 3) : Take $v = k^2 R_0(\lambda + i0)\, m u^{in}$.

3) $\Rightarrow$ 2) : Define $v(x) = 0$ in $\mathbb{R}^n \setminus D$.

Then v is outgoing

$$(\Delta + k^2)\,v = -k^2 m u^{in} \quad \text{in } \mathbb{R}^n$$

$$\Rightarrow \quad v = u^s = k^2 R_0(\lambda + i0)\, m u^{in}.$$

Similarly 4) $\Leftrightarrow$ 5).

1) $\Leftrightarrow$ 2) : Since we have a unique decomp.

$$u^{in} = u^0 + u^s \quad \Rightarrow$$

$$u^s = u^{in} - u^0$$

$\square$

16.2 Discreteness of TE's

We assume here that $m \in C^2(D)$

$$|m| > c_0 > 0 \text{ in } \overline{D} \quad \text{and}$$

$$m = 0 \quad \text{in } \mathbb{R}^3 \setminus D$$

where D is smooth domain in $\mathbb{R}^n$.

16.4 Theorem $\lambda > 0$ is a TE, iff $\exists u \in H_0^2$ satisfying

$$(*) \qquad (P^m - \lambda) \frac{1}{m} (P^0 - \lambda) u = 0$$

Proof ~~We show that (*) $\Leftrightarrow$ 2/ in Th 16.3~~

Note (*) $\Leftrightarrow$

$$(\Delta + k^2(1+m)) \frac{1}{m} (\Delta + k^2) u = 0$$

Assume $\lambda > 0$ is a TE. Then $\exists (v,w) \neq 0$

$$\begin{cases} (\Delta + k^2) v = 0 & \text{in } D \\ (\Delta + k^2(1+m)) w = 0 & \text{in } D \\ v - w \in H_0^2(D) \end{cases}$$

Define $u = v - w \Rightarrow$

$$(\Delta + k^2) u = -(\Delta + k^2) w = k^2 m w$$

$$\Rightarrow \quad (\Delta + k^2(1+m)) \frac{1}{m} (\Delta + k^2) u =$$

$$= (\Delta + k^2(1+m)) k^2 w = 0$$

Assume next (*): We claim

(1) $$(P^m-\lambda)\frac{1}{m}(P^0-\lambda) = (P^0-\lambda)\frac{1}{m}(P^m-\lambda)$$

Note $V = -k^2 m$

$$(P^m-\lambda)\frac{1}{V}(P_0-\lambda)u - (P_0-\lambda)\frac{1}{V}(P^m-\lambda)u$$

$$(P^0+V-\lambda)\frac{1}{V}(P_0-\lambda)u - (P_0-\lambda)\frac{1}{V}(P_0+V-\lambda)u$$

$$= \frac{V}{V}(P_0-\lambda)u - (P_0-\lambda)\frac{V}{V}u = 0$$

proving (1). Now if $P = P^m$ and

$$(P-\lambda)\underbrace{\frac{1}{V}(P_0-\lambda)u}_{w} = (P_0-\lambda)\underbrace{\frac{1}{V}(P-\lambda)u}_{v} = 0$$

Now

$$(P-\lambda)w = 0$$

$$(P_0-\lambda)v = 0$$

and

$$v - w = \frac{1}{V}(P-P_0)u = u \in H_0^2(D)$$

▨

We will study the operator family

$$T_\tau = (P^m-\tau)\frac{1}{m}(P^0-\tau)$$

$$= (\Delta+\tau(1+m))\frac{1}{m}(\Delta+\tau)$$

1.5 Lemma

(i) $T_\tau = \Delta \frac{1}{m} \Delta + \tau\left(\Delta \frac{1}{m} + \left(1 + \frac{1}{m}\right)\Delta\right) + \tau^2\left(1 + \frac{1}{m}\right)$

(ii) $T_\tau = (\Delta + \tau) \frac{1}{m} (\Delta + \tau) + \tau(\Delta + \tau)$

(iii) $T_\tau = (\Delta + \tau(1+m)) \frac{1}{m} (\Delta + \tau(1+m)) - \tau(\Delta + \tau(1+m))$

Proof □

16.6 Lemma For $\tau \geq 0$, T_τ, with domain $H_0^2(D) \cap H^4(D)$ is self-adjoint in $L^2(D)$. T_τ is a rel. compact perturbation of $\Delta \frac{1}{m} \Delta$ with same domain. The associated quad. form t_τ, with form domain $H_0^2(D)$, is given by

$$t_\tau(u) = \int_D \frac{1}{m} |(\Delta + \tau)u|^2 - \tau \int_D |\nabla u|^2 + \tau^2 \int_D |u|^2$$

Proof By Lemma (16.5) (ii) we have for $v \in H_0^2(D)$

$$(T_\tau u, v) = \int \bar v (\Delta + \tau) \frac{1}{m} (\Delta + \tau) u + \tau \int \bar v (\Delta + \tau) u$$

which shows that T_τ is symmetric, and that

$$(T_\tau u, u) = t_\tau(u)$$

$$\Big[= \int (\Delta + \tau) \bar v \frac{1}{m} (\Delta + \tau) u - \tau \int \nabla \bar v \cdot \nabla u + \tau^2 \int \bar v u$$

$\mathcal{D}(T_\varepsilon^*) - \mathcal{D}(T_\varepsilon) : \supset$

$$0 = (\Delta v, u)_{H^2} = \overset{0}{(\Delta v, u)} + \overset{0}{(\Delta^2 v, \Delta u)} + (\nabla \Delta v, \nabla u) = \int \frac{\partial u}{\partial \nu} \Delta v$$

$\forall\ u \in H_0^2 \cap H^4$.

Thus it is enough to show

(*) $\Delta : H_0^2 \cap H^4 \to H^1$ has dense range

But Δ as unbounded op. is symmetric on (*), and injective hence (*) holds.

Thus $\frac{\partial u}{\partial \nu} = 0 \Rightarrow u \equiv 0$. Hence (2) holds.

$\square$

$T = \Delta^2$, $D(T) = H^2_0 \cap H^4$

$u \in D(T^*) \Rightarrow \exists w \in L^2$ s.t. $\forall v \in H^2_0 \cap H^4$

$(u, Tv) = (w, v) \Leftrightarrow (u, \Delta^2 v) = (w, v)$

$\Rightarrow (\Delta^2 u, \varphi) = (w, \varphi)$, $\forall \varphi \in C^\infty_0$

$\Rightarrow \Delta^2 u = w \in L^2 \Rightarrow u \in H^4$

But now $\forall v \in D(T)$

$$(u, \Delta^2 v) = (w, v) = (\Delta^2 u, v) = (\Delta^2 u, \Delta v)$$

$\Rightarrow$ if $f = \Delta v$,

(1) $(u, \Delta f) = (\Delta u, f) \Leftrightarrow \int_{\partial D} \left(\frac{\partial u}{\partial \nu} f - u \frac{\partial f}{\partial \nu}\right) ds = 0$

If

(2) $\Delta : H^4 \cap H^2_0 \to H^2$

has a dense range, then (1) $\Rightarrow u = \frac{\partial u}{\partial \nu} = 0$

$\Rightarrow D(T) = D(T^*)$

To show (2), assume

$u \perp_{H^2} \Delta v$, $v \in H^2_0 \cap H^4$

$\Rightarrow 0 = (\Delta \varphi, u)_{H^2} = (\varphi, \Delta u)_{H^2}$ $\forall \varphi \in C^\infty_0$

Denote $f = \Delta u$. Now

$0 = (f, \varphi)_{H^2} = (f, \varphi) + (\Delta f, \Delta \varphi) = (f, \varphi + \Delta^2 \varphi)$

$\forall \varphi \in C^\infty_0$, But the symm. operator $I + \Delta^2$ is injective $\Rightarrow$ dense range $\Rightarrow f = 0 \Rightarrow$ u is harmonic.

$\Rightarrow$

17 EXISTENCE OF TE's

Assumption $|m| > \delta$, $m \in H^2$

17.1 Lemma (i) If

$$\operatorname{sgn}(m) \inf_{u \in H_0^2} \frac{l_\tau(u)}{\|u\|^2} > 0 \qquad (1)$$

then τ is not TE.

(ii) If $\exists\ u \in H_0^2(D)$ s.t.

$$\operatorname{sgn}(m) \frac{l_\tau(u)}{\|u\|^2} \leq 0 \qquad (2)$$

then $\exists$ a TE $\tau^* \in (0, \tau]$

Proof We may assume that $m > \delta > 0$.

i) T_τ is rel. comp. perturbation of $\Delta \frac{1}{m} \Delta = T_0$. T_0 has a compact resolvent since

$$(\varphi, T_0 \varphi) = \int \Delta\varphi \frac{1}{m} \Delta\bar{\varphi} \geq c \|\Delta\varphi\|^2$$

Thus T_0 has a discrete spectrum in $(0, \infty)$ $\Rightarrow$ T_τ has discrete spectrum and it's eigenvalues are real and dep. cont. on τ.

By (1) $\sigma(T_\tau) \subset (0, \infty)$ $\Rightarrow$ T_τ is injective $\Rightarrow$ τ is not a TE.

(ii) Hypothesis (2) $\Rightarrow$ The smallest eigenvalue $\lambda_0(T_\tau) \leq 0$. But T_0 is posit. def.

$\Rightarrow \lambda_0(T_0) > 0$

Cont of $\tau \mapsto \lambda_0(T_\tau)$ implies

$\exists\ \tau^* \in (0, \tau]$ s.t. $\lambda_0(T_{\tau^*}) = 0$

$\Rightarrow \exists\ \underset{\neq 0}{u} \in H_0^2(D)$ s.t. $T_{\tau^*} u = 0$.

▨

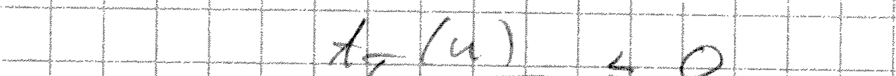
For simplicity we ass. $m \geq \delta > 0$.

<u>17.2 Lemma</u> If $\exists\ \tau > 0$ and $V^p \subset H_0^2(D)$, $\dim V^p = p$ s.t.

$$\frac{t_\tau(u)}{\|u\|^2} \leq 0$$

$\forall\ u \in V^p$, then $\exists\ p$ trans. eigenvalues (count. multipl.) in $(0, \tau]$.

<u>Proof</u> Assumption $\Rightarrow \exists\ p$ negat. eigenvalues of T_τ.



▨

Let $\lambda_0(D)$ be the smallest eigenvalue of Dirichlet Δ and $\mu_0(D)$ the smallest Dirichlet eigenvalue of $\Delta^2 \Rightarrow \exists u, v$

$$\begin{cases} \Delta u = \lambda_0 u, & u \neq 0 \\ u|_{\partial D} = 0 \end{cases}$$

$$\begin{cases} \Delta^2 v = \mu_0 v, & v \neq 0 \\ v|_{\partial D} = 0 \\ \frac{\partial}{\partial \nu} v\Big|_{\partial D} = 0 \end{cases}$$

<u>17.3 Lemma</u>

(i) $$\lambda_0(D) = \inf_{u \in H^1_0(D)} \left(\frac{\|\nabla u\|}{\|u\|} \right) = \inf_{u \in H^2_0(D)} \left(\frac{\|\nabla u\|}{\|u\|} \right)^2$$

(ii) $$\mu_0(D) = \inf_{u \in H^2_0(D)} \left(\frac{\|\Delta u\|}{\|u\|} \right)^2 \geq \inf_{u \in H^1_0 \cap H^2} \frac{\|\Delta u\|^2}{\|u\|} = \lambda_0^2$$

(iii) If $u \in H^2_0(D)$

$$\lambda_0(D) \leq \left(\frac{\|\nabla u\|}{\|u\|} \right)^2 \leq \frac{\|\Delta u\|}{\|u\|}$$

(i) First equality:

$$\Delta_D u = \sum \lambda_i \, u_i \otimes u_i \quad , \quad u_i \text{ ON basis in } L^2$$

$$\Delta u_i = \lambda_i u_i \quad , \quad u_i \in H_0^1(D) \quad , \quad i = 0, 1, \dots$$

Let $u \in H_0^1(D)$, $u = \sum a_i u_i$. Then

$$\int |\nabla u|^2 = (\nabla u, \nabla \bar{u}) = (\Delta u, u)$$

$$= (\Delta \sum a_i u_i , \sum a_k u_k) = (\sum a_i \lambda_i u_i , \sum a_k u_k)$$

$$= \sum_{i=0}^{\infty} \lambda_i |a_i|^2 \geq \lambda_0 \sum_{i=0}^{\infty} |a_i|^2 = \lambda_0 \|u\|^2$$

Thus

$$\lambda_0 \leq \frac{\|\nabla u\|^2}{\|u\|^2} \quad , \quad \forall \, u \in H_0^1(D) .$$

and

$$\|\nabla u_0\|^2 = (\Delta u_0, u_0) = \lambda_0 \|u_0\|^2$$

Second equality: $H_0^2(D) \subset H_0^1(D)$ is dense, since $C_0^\infty(D)$ is dense in $H_0^1(D)$.

(ii) As first inequality in (i), only Δ_D replaced for Δ_D^2.

Note $H_0^2(D) \subset H_0^1 \cap H^2$

To understand the last equality, note

$$\mathcal{D}(\Delta_D^2) = \{ u \in \mathcal{D}(\Delta_D) \mid \Delta_D u \in \mathcal{D}(\Delta_D) \}$$

$$= \{ u \in H_0^1(D) \, ; \, \Delta u \in H_0^1(D) \} \subset H_0^1 \cap H^2$$

If $\Delta_D u_0 = \lambda_0 u_0$ then $\Delta^2 u_0 = \lambda_0^2 u_0$.

i.e. $u_0 \in H^1_0$, $\Delta u_0 \in H^1_0 \Rightarrow$

$$(\Delta_D)^2 u_0 = \lambda_0^2 u_0$$

i.e. λ_0^2 is the smallest eigenvalue of $(\Delta_D)^2$.

Finally, for $u \in H^2 \cap H^1_0$, $u = \sum a_i u_i$

$$(\Delta u, \Delta u) = \sum \lambda_i^2 |a_i|^2 \geq \lambda_0^2 \sum |a_i|^2 = \lambda_0^2 \|u\|^2$$

Hence, if u is minimizing

$$\frac{\|\Delta u\|^2}{\|u\|^2}$$

then $u = c\,u_0 \Rightarrow \Delta u \in H^1_0$

(iii) The first inequality is trivial and

$$\frac{(\nabla u, \nabla u)}{\|u\|} = \frac{(\Delta u, u)}{\|u\|^2} \leq \frac{\|\Delta u\|\,\|u\|}{\|u\|^2} \leq \frac{\|\Delta u\|}{\|u\|}$$

$\square$

<u>17.4 Theorem</u> The set of TE's is discrete

<u>Proof</u> We will write

$$T_E = \Delta \frac{1}{m} \Delta - \tau \Delta \left(1 + \frac{1}{m}\right) + \frac{\tau}{m} \Delta + \tau^2 \left(1 + \frac{1}{m}\right)$$

The claim is based on analytic Fredholm theory. It says: If $\Omega \subset \mathbb{C}$

$F : \Omega \longrightarrow L(H_1, H_2)$, H_i Hilbert spaces is analytic, $F(z)$ is Fredholm for $\forall z \in \Omega$ and $F(z_0)$ is invertible, $z_0 \in \Omega$, then $F(z)$ is invertible except a discrete set of values of z.

The claim will follow if we prove

(i) $T_0 : H_0^2(D) \to H^{-2}(D)$ is invertible

(ii) T_τ is analytic for $\tau \in \mathbb{C}$

(iii) $T_\tau - T_0 : H_0^2(D) \to H^{-2}(D)$

Since $\tau \to T_\tau$ is polynomial (ii) is trivial.

(iii) We have

$$T_\tau - T_0 = \tau\, \Delta\left(1 + \frac{1}{m}\right) + \frac{\tau}{m} \Delta + \tau^2 \left(1 + \frac{1}{m}\right)$$

$$= T_1 + T_2 + T_3$$

T_1 is compact since

$$H_0^2 \underset{\text{comp}}{\hookrightarrow} L^2 \xrightarrow[\text{bounded}]{\Delta(1+\frac{1}{m})} H^{-2}$$

T_2 is compact since

$$\frac{1}{m}\Delta : \quad H_0^2 \underset{\text{bounded}}{\xrightarrow{\Delta}} L^2 \underset{\text{bounded}}{\xrightarrow{\frac{1}{m}}} L^2 \underset{\text{comp.}}{\hookrightarrow} H^{-2}$$

T_3 is trivially compact

To prove (i) note for $u \in H_0^2$

$$(u, T_0 u) = \int_D \Delta u \, \frac{1}{m} \, \Delta \bar{u} \ge \inf \frac{1}{m} \|\Delta u\|_{L^2} \|\Delta u\|_{L^2}$$

But

$$\|u\|_{H^2}^2 = \|u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \le \left(\frac{1}{\lambda_0^2} + 1\right) \|\Delta u\|_{L^2}^2$$

by Lemma 17.3 (iii).

Hence

(1) $$(u, T_0 u) \ge \left(\inf \frac{1}{m}\right) \left(\frac{1}{\lambda_0^2} + 1\right)^{-1} \|u\|_{H^2}$$

Since $T_0 : X \to X^*$, $X = H_0^2$ is symmetric bounded operator, (1) shows

T_0 is invertible

and

$$\|T_0^{-1}\| \le \|m\|_{L^\infty} \left(\frac{1}{\lambda_0^2} + 1\right) \qquad \square$$