We also need estimates for the fundamental solutions $\phi_\pm(x,k) = R_0(\lambda \pm i0)1 = \frac{i}{4}\left(\frac{k}{2\pi|x|}\right)^{\frac{n-2}{2}} H^{(1,2)}_{\frac{n-2}{2}}(k|x|)$ $\left(= \frac{e^{\pm ik|x|}}{4\pi|x|}, \text{ when } n=3\right)$.

15.7 Lemma We have for all $k>0$ in $\mathbb{R}^3$

(i) $$\frac{e^{\pm ik|x-y|}}{|x-y|} = \frac{e^{\pm ikr}}{r}\left\{e^{\mp ik\theta\cdot y} + O\left(\frac{1}{|x|}\right)\right\}$$

(ii) $$\frac{\partial}{\partial\nu(y)}\frac{e^{\pm ik|x-y|}}{|x-y|} = \frac{e^{\pm ikr}}{r}\left\{\frac{\partial}{\partial\nu(y)}e^{\mp ik\theta\cdot y} + O\left(\frac{1}{|x|}\right)\right\}$$

where $r=|x|$ and $\theta = \frac{x}{|x|}$. The estimates hold uniformly w.r.t. $y \in K \Subset \mathbb{R}^n$.

Proof We have

$$|x-y|^2 = x^2 - 2x\cdot y + y^2 = |x|^2\left(1 - 2\frac{\hat{x}\cdot y}{|x|} + O\left(\frac{1}{|x|^2}\right)\right)$$

$$\Rightarrow \quad |x-y| = |x|\left(1 - \frac{1}{2}\left(\frac{2\hat{x}\cdot y}{|x|}\right) + O\left(\frac{1}{|x|^2}\right)\right)$$

$$= |x| - \frac{\hat{x}\cdot y}{|x|} + O\left(\frac{1}{|x|^2}\right)$$

$\Rightarrow$ claim. $\square$

Similar asymptotics hold for $H^{(1,2)}_{\frac{n-2}{2}}(k|x|)$. Which yield

15.8 Lemma Assume $f \in B$. The asymptotics of $u^{\pm} = R_0(\lambda \pm i0) f$ is given by

$$u^{\pm}(x) = C_n \frac{k^{\frac{n-3}{2}} e^{\pm ik|x|}}{|x|^{\frac{n-1}{2}}} \hat{f}(\pm k\theta) + u_0^{\pm}$$

where $u_0^{\pm} \in \overset{\circ}{B}{}^*$. Here $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx$.

$$C_n = \frac{1}{4\pi}\left(\frac{1}{2\pi i}\right)^{\frac{n-3}{2}}$$

Proof

We prove the claim only in the case $n = 3$.

$$u^{\pm}(x) = \int \frac{e^{\pm ik|x-y|}}{4\pi|x-y|} f(y)\,dy$$

L 17.7.(i)

$$= \frac{e^{\pm ikr}}{4\pi r} \int \left(\frac{e^{\mp ik\theta\cdot y}}{4\pi} + O\left(\frac{1}{r}\right) \right) f(y)\,dy$$

$$= \frac{e^{\pm ikr}}{4\pi r} \hat{f}(\pm k\theta) + u_0^{\pm}$$

Here

$$|u_0^{\pm}(r\theta)| \le \frac{C}{(1+r)^2} \quad \text{for } f \in L^2_{comp}.$$

Thus if

$$f_n = \chi_{B_n} f \in L^2_{comp}$$

we have

$$\|f_n - f\|_B \to 0 \quad \text{and}$$

$$f_n \in \overset{\circ}{B}{}^*$$

Thus

$$\underbrace{\|u_n^{\pm} - u^{\pm}\|_{B^*}}_{\to 0} + \left\|\frac{e^{\pm ik}}{r}\left(\hat{f}_n(k\theta) - \hat{f}(k\theta)\right)\right\|_{B^*}$$

$$\ge \|u_{n,0}^{\pm} - u_0^{\pm}\|_{B^*}$$

By Theorem 14.1.1, [Hö] the map

$$f \mapsto \hat{f}(k\theta)$$

is bounded (surjective) from $B \to L^2(M_\lambda)$.

Thus $\int | \hat{f}_n(k\theta) - \hat{f}(k\theta) |^2 \, d\theta \to 0$ as $n \to \infty$

$$\Rightarrow \quad \left\| \frac{e^{\pm ikr}}{r} \left(\hat{f}_n(k\theta) - \hat{f}(k\theta) \right) \right\|_{B^*} \to 0$$

$$\Rightarrow \quad \| u^{\pm}_{n,0} - u^{\pm}_0 \|_{B^*} \to 0$$

$$\Rightarrow \quad u^{\pm}_0 \in \overset{\circ}{B}{}^* . \qquad \square$$

<u>Proof of Theorem 15.6:</u>

(i) $\quad u^0 \in B^0 \iff u^0 \in B^*$ and $(\Delta + k^2) u^0 = 0$

$$\Rightarrow \quad u = u^{\pm} = \int e^{\mp ik\theta \cdot x} \, v^{\pm}(\theta) \, d\theta + 0$$

Here

$$v^{\pm} \in L^2(S^{n-1})$$

For $v_j \in C^\infty(S^{n-1})$ we have for

$$u_j = \int_{S^{n-1}} e^{-ik\theta \cdot x} \, v_j(\theta) \, d\theta$$

by (7.7.25) that

$$u_j(r\theta) = \left(\frac{2\pi k}{r} \right)^{\frac{n-1}{2}} \left[v_j(k\theta) e^{-irk} + v_j(-k\theta) e^{irk} \right] + O\left(\frac{1}{r^2}\right)$$

Since $O(\frac{1}{r^2}) \in \mathring{B}^*$ and

$$F : L^2(S^{n-1}) \to B^* \quad \text{is bounded}$$

we get in the limit $j \to \infty$

$$u^+(r\theta) = \left(\frac{2\pi k}{r}\right)^{\frac{n-1}{2}} \left[v_+(\theta) e^{ikr} + v_+(-\theta) e^{-ikr} \right] + u^0$$

where $u^0 \in \mathring{B}^*$.

ii) By definition, $u^+ \in B^* \Leftrightarrow u \in B^*$ and

$$u^+ = R_0(\lambda + i0) f \quad , f \in B$$

Lemma 15. 8 $\Rightarrow$

$$u^+ = C_n \frac{1}{k} \left(\frac{k}{r}\right)^{\frac{n-1}{2}} e^{ikr} \hat{f}(k\theta) + u_0 \quad \in \mathring{B}^*$$

iii) Since

$$u = u_0 - R(\lambda + i0) V u \quad , u_0 = \int e^{-ik\theta x} v_+(\theta) d\theta$$

the claim follows from (i) and (ii).

$\square$

Remark If

$$u = u_0 + R_0(\lambda + i0) V u$$

then

$$\mu^m(\theta) = (2\pi k)^{\frac{n-1}{2}} v_+(\theta)$$

when $$u^0 = \int_{S^{n-1}} e^{-ik\theta \cdot x} v_+(\theta) d\theta$$ and

$$\mu^{m}(\theta) = C_n k^{\frac{n-3}{2}} \int e^{-ix \cdot k\theta} V(x) u(x) dx$$

$s^m(\theta)$ is called the <u>scattering amplitude</u>.

We define three maps

$$b^0 : B^0 \to L^2(S^{n-1})$$

$$b^m : B^m \to L^2(S^{n-1}) \quad \text{and}$$

$$b^+ : B^+ \to L^2(S^{n-1})$$

by

$$b^0 : u^0 \mapsto \mu^0$$

$$b^m : u^m \mapsto \mu^m \quad \text{and}$$

$$b^+ : u^+ \mapsto \mu^+$$

<u>15.9 Theorem</u> b^0 and b^m are isomorphisms and b^+ is surjective. Moreover

$$(1) \qquad \operatorname{Ker} b^+ = \{ R(\lambda+i0) f \mid \hat{f}|_{kS^{n-1}} = 0 \}$$

<u>Proof</u> Rellich $\Rightarrow$ b^0 and b^m are injective.

Since

$$u^0 = c \int e^{+ik\theta\cdot x} \mu_0(\theta)\, d\theta$$

the operator b^0 is also surjective.

Since for any u^0

$$u = u_0 + R(\lambda+i0) V u$$

has a unique solution in $B^*_{p_0}$, we conclude that b_λ^m is also surjective.

How to prove b^+ is surjective?

$$b^+ : u^+ \to \mu^+$$

$$u^+(x) = R_0(\lambda + i0) f = \frac{e^{i k r}}{r^{\frac{n-1}{2}}} \mu_+(\theta) + \mathring{B}^*$$

where

(A) $$\mu_+(\theta) = c \hat{f}(k\theta) \quad , \quad c = k^{\frac{n-3}{2}} c_n$$

$$f = V u^m \in B$$

Now claim follows from the fact that

$$f \mapsto \hat{f}(k\theta) \quad \text{is surjective from } B \to L^2(S^{n-1})$$

(This is Theorem 14.1.1 [Hö]).

Finally (2) $\to$ (1) ■

Recall

$$S : u^0 \mapsto u^m \quad , \quad \begin{array}{l} u^0 \in B^0 \\ u^m \in B^m \end{array}$$

$$\Sigma : v_- \mapsto v_+ \quad , \text{ where}$$

$$u^m = u_\pm + R(\lambda \mp i0) V u^m \quad , \quad \widehat{u_\pm} = v_\pm \, dS$$

Finally we have the operator

$$A : \mu^m \to \gamma$$

The next lemma establishes the connection between these

15.10 Lemma We have

(i) $A = 1 - \Sigma$ and

(ii) $\forall\, u^0, w^0 \in B$ with far fields μ_0 and ν_0 we have

$$\int_{\mathbb{R}^n} u^0\, m\, S w^0 = -2ik \int_{S^{n-1}} \bar{\mu}^0 A \nu^0 , \quad c_n$$

Proof (i) By Lemma 15.8

$$R(\lambda \pm i0) f \approx c \frac{e^{\pm ik|x|}}{|x|^{\frac{n-1}{2}}} \hat{f}(\pm k\theta)$$

Since

$$u = u^0_\pm + R(\lambda \mp i0) V u$$

$$\hat{u}^0_\pm (k\theta) = v^\pm (k\theta) ds$$

T. 15.6 (i) $\Rightarrow$
$$u^0_\pm \approx v^\pm (k\theta) \frac{e^{ikr}}{r^{\frac{n-1}{2}}} + v^\pm(-k\theta) \frac{e^{-ikr}}{r^{\frac{n-1}{2}}}$$

$\Rightarrow$
$$u^m \approx (v^+(k\theta) + \hat{f}(k\theta)) \frac{e^{ikr}}{r^{\frac{n-1}{2}}} + v^+(-k\theta) \frac{e^{-ikr}}{r^{\frac{n-1}{2}}}$$

$$\approx v^-(k\theta) \frac{e^{ikr}}{r^{\frac{n-1}{2}}} + (v^-(-k\theta) + \hat{f}(k\theta)) \frac{e^{-ikr}}{r^{\frac{n-1}{2}}}$$

Thus $\mu^m(\theta) = v^+(k\theta)$, $\nu(\theta) = \hat{f}(k\theta)$

and

$$v^-(\omega) = \mu^m(\theta) + \gamma(\theta)$$

Thus

$$\Sigma \mu^m = \Sigma v^+ = v^- = \mu^m + \gamma = (1+A)\mu^m$$

which proves

$$\Sigma = 1 + A$$

(ii) By writing $W^m = W^0 + W^+$ we have

$$\int_{\mathbb{R}^n} \bar{u}_0 k^2 m S W^0 = \int_{\mathbb{R}^n} \bar{u}_0 k^2 m W^m$$

$$= -\int \bar{u}_0 (\Delta + k^2) W^m = -\int_{\mathbb{R}^n} \bar{u}_0 (\Delta + k^2) W^+$$

$$= \lim_{R\to\infty} -\int_{B_R} \bar{u}^0 (\Delta + k^2) W^+ =$$

$$= \lim_{R\to\infty} \int_{\partial B_R} \frac{\partial \bar{u}^0}{\partial \nu} W^+ - \bar{u}_0 \frac{\partial W^+}{\partial \nu} = \lim_{R\to\infty} \int_{\partial B_R} -2ik \frac{\mu_0(\theta) A \nu^0(\theta)}{R^{n-1}}$$

$$= -2ik \int \mu^0 A \nu^0$$

By Theorem 15.6

$\boxtimes$

Define

$$S^+ : B^0 \to B^{0*}$$

by

$$S^+ : w^0 \mapsto -\frac{1}{2ik} \langle m S w^0, \cdot \rangle$$

15.11 Theorem

$$A = \mathcal{H}^* S^+ \mathcal{H}$$

where $\mathcal{H}$ is the Herglotz operator

$$u^0 = \mathcal{H}\mu_0 = \int_{S^{n-1}} e^{ik\theta\cdot x} \mu(\theta)\, d\theta$$

Proof

$$\langle \mathcal{H}^* S^+ \mathcal{H} \nu^0, \overline{\mu_0} \rangle = \langle S^+ \mathcal{H} \nu^0, \overline{\mathcal{H}\mu_0} \rangle$$

$$= -\frac{1}{2ik} \int_{\mathbb{R}^n} \overline{\mathcal{H}\mu_0}\, m S \mathcal{H} \nu^0$$

$$= -\frac{1}{2ik} \int_{\mathbb{R}^n} \overline{u_0}\, m S w^0 = \int_{S^{n-1}} \mu^0 A \nu^0$$

$$= \langle A\nu^0, \overline{\mu_0} \rangle$$

$\square$

Remark An element $\ell \in B^{0*}$ can be interpreted as follows: $u^0 \in B^0 \Rightarrow u^0 = \mathcal{H}\mu_0$. (Dense)

$\mathcal{H} : L^2 \to B^0$ and $\mathcal{H}^* : B^{0*} \to L^2$ and by denoting $v^0 = \mathcal{H}^* l$ we have for any $\eta_0 \in L^2(S^{n-1})$

$$\langle v^0, \eta^0 \rangle_{S^{n-1}} = \langle \mathcal{H}^* l, \eta^0 \rangle = \langle l, \mathcal{H} \eta_0 \rangle$$

Thus

$$\boxed{\langle l, u_0 \rangle = \langle v^0, \eta^0 \rangle \quad , \text{ if } u^0 = \mathcal{H} \eta^0 .}$$

<u>16. A Generalized Scattering Operator and interior transmission problem</u>

In Section 15 we used sources

$$u^0 = \mathcal{H} \mu_0$$

to illuminate the scatterer m.

Other sources might be supported in a compact set D.

Let $m \in L^\infty$, $\operatorname{supp} m \subset D$, and $u^0 \in L^2(D)$.

By Theorem 15.3 $\exists !\, u^+ \in B^+$ solving

(1) $\quad (\Delta + k^2(1+m)) u^+ = -k^2 m u^0$

$\Rightarrow$

$$u^m = u^0 + u^+ \in L^2(D)$$

Note that u^+ is outgoing in the whole $\mathbb{R}^n$.

We want to extend the scattering map S to such sources.

Let

$$B_D^0 = \{ w \in L^2(D) \mid (\Delta + k^2) w = 0 \}$$

$$B_D^m = \{ w \in L^2(D) \mid (\Delta + k^2(1+m)) w = 0 \}$$

We define

$$S_D : B_D^0 \to B_D^m \quad , \quad u^0 \mapsto u^m \quad \text{and}$$

$$S_D^+ : B_D^0 \to B_D^{0\,*}$$

$$S_D^+ : w^0 \mapsto \frac{1}{2i} \langle m\, S_D w^0, \cdot \rangle$$

Clearly, since the right hand side is independent how u^0 is defined outside of supp m we have for $u^0, w^0 \in B^0$

$$S_D u^0 = S u^0 |_D$$

$$\langle S_D^+ u^0, w^0 \rangle = \langle S^+ u^0, w^0 \rangle$$

The interior transmission problem (ITP)

From now on denote

$$P^0 = (\Delta + k^2)$$

$$P^m = (\Delta + k(1+m))$$

$$u \in H^k(D) \iff \|\partial^\alpha u\|_{L^2} < \infty \quad \forall |\alpha| \leq k$$

$\mathring{H}^k(D)$ is the closure of C_c^∞ in $H^k(D)$

16.1. Definition k is a D-transmission eigenvalue for $m \in L^\infty(D)$, if $\exists$ $u^0 \in B_D^0$ and $u^m \in B_D^m$ s.t.

$$0 \neq u^0 - u^m \in \mathring{H}^2(D)$$

We also say that k is a transmission eigenvalue (TE) if $\exists$ $u_0 \neq 0$ s.t. the scattered field is both incoming and outgoing

$$u^m = u^0 + \underbrace{R(\lambda + i0)\, V u^m}_{=: u^s} \qquad V = k^2 m$$

Since supp V is compact we have

$$u^s \equiv 0 \quad \text{outside supp } V$$

Now if supp $V \subset D$ we have

$$u^0 \in B_D^0 \, , \quad u^m \in B_D^m$$

and

$$u^0 - u^m = u^s$$

vanishes in the neighborhood of ∂D. Then $u^0 - u^m \in \mathring{H}_0^2(D)$ and the interior transmission problem

$$(\Delta + k^2)\, v = 0 \quad \text{in } D$$
$$(\Delta + k^2(1+m))\, w = 0 \quad \text{in } D$$

$$v = w \, , \quad \frac{\partial}{\partial \nu} v = \frac{\partial}{\partial \nu} w \quad \text{on } \partial D$$

has a non-trivial solution $v = u^0$ and $w = u^m$.