VIIKKO 3, SYKSY 2008

15.2 Lemma (Rellich) Assume that v satisfies $(\Delta + k^2)v = 0$, for $|x| > R_0$ and that $v(x) = o(|x|^{-\frac{n-1}{2}})$. Then

$v(x) = 0$ for $|x| \geq R_0$

Proof Let Δ_S be the Laplace-Beltrami-operator on S^{n-1}. It is enough to show that

$$V(r) = \int_{S^{n-1}} v(r\omega)\varphi(\omega)\,dS(\omega) \equiv 0$$

for each eigenfunction φ on Δ_S.

Recall

$$\Delta = \left(\frac{\partial}{\partial r}\right)^2 + \frac{n-1}{r}\frac{\partial}{\partial r} + r^{-2}\Delta_S$$

Now

$$(\Delta_S + \mu^2)\varphi = 0 \quad , \mu \geq 0$$

Apply $\left(\frac{\partial}{\partial r}\right)^2 + \frac{n-1}{r}\frac{\partial}{\partial r}$ to V (considered to be a radial function in $\mathbb{R}^n$)

$$V''(r) + \frac{n-1}{r}V'(r) = (\Delta - r^{-2}\Delta_S)V$$

$$= \left(-k^2 + \frac{\mu^2}{r^2}\right)V(r).$$

This is a Bessel-equation and

②

and it has two independent solutions (Hankel-functions) $H^{(\nu)}$ with asymptotics

$$H^{(\nu)}(r) = C_\pm \frac{e^{\pm ikr}}{r^{\frac{n-1}{2}}} + o\left(r^{-\left(\frac{n-1}{2}\right)}\right), \quad C_\pm \neq 0$$

By hypothesis

$$\lim_{r\to\infty} r^{\frac{n-1}{2}} V(r) = \lim_{r\to\infty} \int_{S^{n-1}} v(r\omega)\, r^{\frac{n-1}{2}}\, \varphi(\omega)\, d\omega$$

$= 0$ and hence

(1) $$V(r) = o\left(r^{-\left(\frac{n-1}{2}\right)}\right).$$

Now

$$V(r) = A_+ H^{(1)} + A_- H^{(2)}$$

$$= A_+ C_+ \frac{e^{ikr}}{r^{\frac{n-1}{2}}} + A_- C_- \frac{e^{-ikr}}{r^{\frac{n-1}{2}}} + o\left(r^{-\frac{n-1}{2}}\right)$$

This and (1) is possible only if

$$A_+ C_+ = A_- C_- = 0$$

$$\Rightarrow A_+ = A_- = 0 \Rightarrow V \equiv 0.$$

$\boxed{}$

An outgoing solution of

$$(\Delta + k^2) v^+ = f \in L^2_{comp} \subset B$$

is called an <u>outgoing wave</u>. By Lemma 1 it is the unique solution of

$$(\Delta + k^2) v^+ = f$$

$$\text{(SRC)} \qquad \left(\frac{\partial}{\partial r} - ik \right) v^+ = o\left(r^{-\frac{n-1}{2}} \right).$$

We study the potential scattering in the form

$$(\Delta + k^2 + k^2 m) u = 0$$

i.e. we take $m = \frac{V}{k^2}$, $k > 0$

and assume that V is compactly supported.

Note that v^+ is outgoing wave $\Leftrightarrow$

$$(15.1) \qquad (\Delta + k^2 (1+m)) w^+ = g$$

where $g = f + k^2 m v^+ \in L^2_{comp}$ and $w^+ = v^+$.

<u>15.3 Theorem</u> $\forall\, g \in L^2_{comp}$ (15.1) has a unique outgoing solution w^+ satisfying

$$\| w^+ \|_{B^*} \leq C \| g \|_B \quad , \quad k \geq \delta > 0$$

where C depends only on δ.

<u>Proof</u> <u>Ex 1</u>

We will call any solution u^0 of

$$(\Delta + k^2) u^0 = 0$$

$$u^0 \in B^*$$

an <u>incident wave</u> or <u>free wave</u>

<u>Ex 2</u> $u^0 = \int_{M_\lambda} e^{ix\cdot\xi} v_\pm(\xi)\, dS = \int_{S^{n-1}} e^{k x\cdot\theta} g_\pm(\theta)\, d\theta$

Similarly any solution U_m of

$$(\Delta + k^2(1+m)) U_m = 0$$

$$U_m \in B^*$$

is called a <u>total wave</u>.

<u>15.4 Theorem</u> Every total wave v^m has a unique decomp.

(i) $\quad v^m = v^0 + v^+$

where v^0 is free wave and v^+ is outgoi

Every free wave u_0 has a uniq. decomp

(ii) $\quad u_0 = u^m - u^+$

where u^m is a total wave and u^+ outgoing.