By definition of $f_{\pm}$ and Thm. 14.5.4 we have

$$(*) \qquad f = f_{\lambda \pm i0} + V R_0(\lambda \pm i0) f_{\lambda \pm i0}$$

Recall the formula (14.3.15)

$$(1) \qquad \pm \frac{1}{(2\pi)^{n-1}} \int_{M_\lambda} |\hat{f}|^2 \frac{ds}{|P_0'|} = 2\,\mathrm{Im}\,(R_0(\lambda \pm i0) f, f)$$

By $(*)$ $R(\lambda \pm i0) f = R_0(\lambda \pm i0) f_{\lambda \pm i0}$ and thus

$$(2) \quad 2\,\mathrm{Im}\,(R(\lambda \pm i0) f, f) = 2\,\mathrm{Im}\,(R_0(\lambda \pm i0) f_{\lambda \pm i0}, f_{\lambda \pm i0})$$

since $R^*(z) = R(\bar{z})$ and $R_0^*(z) = R_0(\bar{z})$.

Thus by (1) and (2)

$$(3) \quad 2\,\mathrm{Im}\,(R_0(\lambda \pm i0) f, f) = \pm \frac{1}{(2\pi)^{n-1}} \int_{M_\lambda} |\hat{f}_{\lambda \pm i0}|^2 \frac{ds}{|P_0'|}$$

and hence by (14.6.1)

$$\int \chi(\lambda) (dE_\lambda f, f) = \frac{1}{(2\pi)^n} \int \chi_\lambda \int_{M_\lambda} |\hat{f}_{\lambda \pm i0}|^2 \frac{ds}{|P_0'|} \, d\lambda$$

$\square$

<u>Corollary</u>

$$(**) \quad \int_{\mathbb{R} \setminus \Lambda} (dE_\lambda f, f) = \frac{1}{(2\pi)^n} \int_{\mathbb{R} \setminus \Lambda} \int_{M_\lambda} |\hat{f}_{\lambda \pm i0}|^2 \frac{ds}{|P_0'|} \, d\lambda$$

<u>Proof</u> $\exists\ \chi_n$, $\mathrm{supp}\, \chi_n \subset \mathbb{R} \setminus \Lambda$ s.t.

$\chi_n \nearrow \chi_{\mathbb{R} \setminus \Lambda}$. The claim follows from monot. convergence.

Note that $\hat{f}_{\lambda\pm i0}(\xi)$ is only defined almost everywhere on M_λ. We hence need

<u>14.6.2 Lemma</u> For $f \in B$, there exist measurable

$$F_\pm f : \mathbb{R}^n \longrightarrow \mathbb{C} \quad \text{s.t.}$$

$$F_\pm f(\xi) = \hat{f}_{\lambda\pm i0}(\xi), \text{ for a.e. } \xi \in M_\lambda .$$

The functions $F_\pm f$ are uniquely defined a.e. in $\mathbb{R}^n$.

<u>Proof</u> We recall that

(1) $$|P_0^{-1}\tilde{\Lambda}| = 0 .$$

If $F_1 f$ and $F_2 f$ would be two functions of the lemma and $g = F_1 f - F_2 f$, we would have

$$g(\xi) = F_1 f(\xi) - F_2 f(\xi) = \hat{f}_{\lambda\pm i0}(\xi) - \hat{f}_{\lambda\pm i0}(\xi)$$
$$= 0 , \text{ for all } \lambda \in \mathbb{R}\setminus\tilde{\Lambda} \text{ and } \xi \in M_\lambda$$

Thus $g(\xi) \neq 0 \Rightarrow P_0(\xi) \in \tilde{\Lambda}$

implying $|\{\xi \mid g(\xi) \neq 0\}| = 0$. This proves the uniqueness.

Now assume $\lambda \in \mathbb{R}\setminus\tilde{\Lambda}$ and choose $g_\lambda^j \in \mathcal{F}(C_0^\infty)$ with

(2) $$\| g_\lambda^j - \hat{f}_{\lambda+i0} \|_B < 2^{-j}$$

Note $\mathcal{F}^{-1}(C_0^\infty)$ is dense in B!

Since $f_{\lambda+i0}$ is cont. w.r.t. λ we can (after some manipulation) make $\hat{g}^j_\lambda$ cont w.r.t. λ also. From (2) it follows

$$(3) \qquad \int_{M_\lambda} |\hat{f}_{\lambda+i0}(\xi) - \hat{g}^j_\lambda(\xi)|^2 \, ds(\xi) \le C(\lambda) 2^{-j}$$

where $C(\lambda)$ is cont in $\lambda \in \mathbb{R}\setminus\tilde{\Lambda}$.

Assume $K \subset \mathbb{R}^n \setminus P_0^{-1}(\tilde{\Lambda})$ is compact. The

$$\int_K |\hat{g}^{j+1}_{P_0(\xi)}(\xi) - \hat{g}^j_{P_0(\xi)}(\xi)|^2 d\xi \le C_K 2^{-j}$$

$$\Rightarrow \exists \; F_+ f(\xi) = \lim_{j\to\infty} \hat{g}^j_{P_0(\xi)}(\xi) \quad \text{for a.e. } \xi \in$$

(Define $F_+ f(\xi) = 0$ if $\nexists$ lim or $\lambda \in \tilde{\Lambda}$)

Note that $P_0(\xi)$ is a local coordinate.

Clearly, F_+ is measurable, $F_+ \in L^2(K)$ $\forall K \subset \mathbb{R}^n \setminus P_0^{-1}(\tilde{\Lambda})$ compact. Moreover by (3)

$$\hat{f}_{\lambda+i0}(\xi) \overset{a.e.}{=} F_+ f(\xi), \quad \text{for } \xi \in M_\lambda$$

$\square$

<u>14.6.3 Definition</u> For $f \in B$ the functions

$$F_\pm f(\xi) = F(1+VR_0(\lambda \pm i0))^{-1} f(\xi)$$

defined a.e. in M_λ for all $\lambda \in \mathbb{R}$ are called the <u>distorted Fourier-transform</u> of f.

Note that

$$F_\pm f\big|_{M_\lambda} \in L^2(M_\lambda) \ , \ \forall \lambda \in \mathbb{R}.$$

We will use the notation

(14.6.4) $$E^d = \int_{\tilde{\Lambda}} dE_\lambda \ , \ E^c = \int_{\mathbb{R}\setminus\tilde{\Lambda}} dE_\lambda$$

Since $\tilde{\Lambda}$ is countable

$$\tilde{\Lambda} = \bigcup_{j=1}^{\infty} \{\lambda_j\}$$

and by denoting $E_j = E_{\lambda_j}$ we have

$$E^d = \sum_{j=1}^{\infty} E_{\lambda_j} \quad ; \quad E_{\lambda_j} u = \int_{\lambda_j} dE_\lambda u$$

and $u \in E_{\lambda_j} L^2 \Rightarrow E_{\lambda_j} u = u$

$$Hu = \int \lambda \, dE_\lambda E_{\lambda_j} u = \int_{\tilde{\Lambda}} \lambda \, dE_\lambda E_{\lambda_j} u$$

$$+ \underbrace{\int_{\mathbb{R}\setminus\tilde{\Lambda}} \lambda \, dE_\lambda E_{\lambda_j} u}_{=0} = \sum_k \lambda_k E_{\lambda_k} E_{\lambda_j} u$$

$$= \lambda_j u.$$

Thus u is eigenvalue of H and $E^d L^2$ is spanned by L^2-eigenvalues of H.

Denote $H^c := H\big|_{E^c L^2} : \overbrace{E_c L^2}^{= L^2_c} \to E^c L^2$ and

$$H^d := H : \underbrace{E^d L^2}_{= L^2_d} \to E^d L^2$$

By Corollary 14.5.6 the point spectrum of H^c is empty $\Rightarrow$ H^c only has cont. spectrum

Next we show that the spectrum of H^c is abs. cont.

<u>14.6.4 Theorem</u> $\forall f \in B$,

(14.6.5) $\quad \|E^c f\|_{L^2} = \frac{1}{(2\pi)^n} \int |F_\pm f|^2 \, d\xi$ and

$F_\pm$ has extension $F_\pm : L^2(dx) \to L^2\left(\frac{d\xi}{(2\pi)^n}\right)$

by defining $F_\pm\big|_{E^d L^2} = 0$. $F_\pm$ intertwine H with pointwise multiplication $\mathring{P}_0 := F H_0 F^{-1}$

(14.6.6) $\quad F_\pm e^{itH} = e^{it\mathring{P}_0} F_\pm \quad , t \in \mathbb{R}$

<u>Proof</u> For clarity, $\mathring{P}_0$ is the operator

$$(\mathring{P}_0 u)(\xi) = P_0(\xi)\, u(\xi)$$

First, (14.6.5) is the limit of (14.6.1)'' as $\chi \nearrow 1$ in $\mathbb{R} \setminus \tilde{\Lambda}$

Next we show

(14.6.7) $\quad F_\pm H f = \mathring{P}_0 F_\pm f \quad , \quad f \in S$.

8.10.08