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Mellin operators and pseudodifferential operators on graphs

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Abstract

We demonstrate how Mellin operator calculus can be used to study the mapping properties of the Dirichlet to Neumann map on graphs. These operators naturally arise in photonic crystal theory. The calculus adds rigour to some previously given heuristic arguments.

Introduction

In this paper we explain how a calculus of pseudodifferential operators on graphs can be built using the Mellin transform instead of (or rather together with) the Fourier transform, and briefly apply it to some examples. The need for such operators arises naturally in photonic crystal theory as explained in [2, 4, 6] and [5]. Most of the material is not new; indeed the calculus itself is due to Rempel and Schulze [7], who based it on the earlier work of Boutet de Monvel [1] and Melrose [9], among others, and the examples are from the paper of [7]. The paper is not self-contained, and for some of the proofs the reader is referred to the works of Rempel and Schulze [10, 11].

The structure of the paper is as follows.

In the first section we introduce the Mellin transform and a scale of function spaces on the half-line $\mathbb{R}_+$ which are an inherent part of the calculus. These spaces generalize the classical Sobolev spaces, but allow a more detailed description of the (singular) behaviour of distributions near the origin, or infinity, if needed. Localizing, we can then lift these spaces to graphs.

The second section is devoted to the description of (classical) pseudodifferential operators on the half-line, and how they can be represented as Mellin operators. We will especially focus on classical layer potential operators, such as the single layer with a Newtonian kernel. A new feature compared to the case without a boundary is that these operators generally produce singularities even for smooth data. To handle these singularities, and also to make the calculus work, one needs a new class of operators, namely the class of Green operators. These operators are classical smoothing operators when restricted to the interior, but at the origin their *conormal order* is non-trivial. So, in this section we explain what the conormal order is, and what the Green operators look like.

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In the third section we describe how one can attach a sequence of conormal symbols to a pseudodifferential operator, which we assume for simplicity to have constant coefficients.

Then, in the fourth section the operator classes are defined, and the symbolic calculus described. We also indicate how elliptic symbols give rise to Fredholm operators.

In the last section we apply the above calculus to analyse the Dirichlet to Neumann operator on a compact graph, and analyse one example in detail.

Most of the notation that we employ is standard, and what is not will be introduced when needed.

1. Function spaces

In this section we recall basic properties of the Mellin transform and introduce some useful function spaces. The original paper of Mellin is [8]; for more facts on the Mellin transform see for example [13, 10] and [11].

For a function $u \in C_0^\infty(\mathbb{R}_+)$, its Mellin transform $\tilde{u}$ is defined for all $z \in \mathbb{C}$ by[1]

$$\tilde{u}(z) = \int_0^\infty t^{z-1} u(t)\,\mathrm{d}t.$$

Whenever it simplifies the notation we also denote this by $Mu(z)$. The reason for the exponent $z-1$ of t above is simple: it makes functions $z \mapsto a^{-z}$ characters of translations $t \mapsto at$, $a > 0$; i.e. if $u_a(t) = u(at)$, then $M(u_a)(z) = a^{-z} Mu(z)$. This corresponds to the familiar fact that translations of $\mathbb{R}$ turn into multiplications by exponentials in Fourier variables. Of course, there is a deeper connection between Mellin and Fourier transforms: let $x = \ln t$, and define $v(x) = \mathrm{e}^{x/2} u(\mathrm{e}^x)$; then for the Fourier transform $\hat{v}$ we have

$$\hat{v}(\xi) = \tilde{u}(1/2 + \mathrm{i}\xi).$$

Hence the natural domain for the Mellin variable is the line $\{z; \operatorname{Re} z = 1/2\}$; here $\operatorname{Re} z$ is the real part of the complex number z. Using this relation, one immediately gets several useful properties of the Mellin transform. First of all, it determines an isometric isomorphism $M : L^2(\mathbb{R}_+) \to L^2(\operatorname{Re} z = 1/2)$ with an inverse transform given by

$$M^{-1} f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{-(1/2+\mathrm{i}y)} f(1/2 + \mathrm{i}y)\,\mathrm{d}y.$$

Also, the analogues of the Paley–Wiener theorems hold. For example, a function f defined on the complex plane is a Mellin transform of some $u \in C_0^\infty(\mathbb{R}_+)$ if and only if it is entire, and for any positive integer m satisfies an estimate

$$|v(z)| \leqslant C_m (1 + |z|)^{-m} |a|^{|\operatorname{Re} z|}$$

for some $a > 1$. If the above holds, then actually $\operatorname{supp} u \in [a^{-1}, a]$. All of this can easily be extended to distributions: let $u \in \mathcal{E}'(\bar{\mathbb{R}}_+)$, the space of extendable distributions with compact support in $\bar{\mathbb{R}}_+$. This means that u is a restriction to $\mathbb{R}_+$ of some $v \in \mathcal{E}'(\mathbb{R})$. If one denotes by r_+ the restriction operator, then one simply defines $\tilde{u}(z) = \langle v, r_+ t^{z-1} \rangle$. This is well defined and analytic for all z with $\operatorname{Re} z$ large enough since v has finite order, and is then independent of the choice of v. The Mellin transform thus defined has the following useful properties:

Proposition 1.1. *For all $u \in \mathcal{E}'(\bar{\mathbb{R}}_+)$ one has the following three relations, whenever* $\operatorname{Re} z$ *is such that both sides are well defined.*

- $M(t\,\mathrm{d}u/\mathrm{d}t)(z) = -z\tilde{u}(z)$.

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[1] Note that we employ a slightly different convention to that in [7]; we chose the one used in [10].

- $M(t^p u)(z) = \tilde{u}(z + p)$ *for all* $p \in \mathbb{C}$.
- $M(\ln tu)(z) = \partial_z \tilde{u}(z)$.

The proofs are obvious. However, there are two crucial observations that one can make from these results. The *conormal derivative* $t\,\mathrm{d}/\mathrm{d}t$ plays the role of the ordinary derivative if one compares Mellin and Fourier transforms; i.e. on the transform side it corresponds to multiplication by the Mellin variable. Secondly, the decay of u at 0 (or ∞) is reflected by the domain of analyticity of the Mellin transform. This may help to motivate the following definition.

Definition 1.2. For any real values of s and γ we define the space $K^{s,\gamma}$ as the closure of $C_0^\infty(\mathbb{R}_+)$ with respect to the Hilbert norm $\|\cdot\|_{s,\gamma}$ given by

$$\|u\|_{s,\gamma}^2 = \int_{\mathrm{Re}\, z=1/2-\gamma} (1 + |z|^2)^s |M(\omega u)(z)|^2\,\mathrm{d}m(z) + \|(1-\omega)u\|_{H^s}^2. \tag{1.1}$$

Here, ω is a smooth cut-off at zero, i.e. $\omega \in C_0^\infty(\bar{\mathbb{R}}_+)$, it is non-negative and equal to one in a neighbourhood of zero, $\|\cdot\|_{H^s}$ is the usual Sobolev norm, and $\mathrm{d}m(z)$ is the Lebesgue measure on the line $\mathrm{Re}\, z = 1/2 - \gamma$.

Let us explain this: first of all, away from the origin we want the function space to coincide with the Sobolev space H^s; this explains the localization with the cut-off. Secondly, near the origin we want to control both the decay at zero, and the smoothness. The decay is controlled by the choice of the line of integration using Cauchy's theorem[2] and the second relation in the proposition above, and the smoothness of (fractional) derivatives is controlled by the weight factor $(1 + |z|^2)^s$. Using another cut-off instead of ω will result in an equivalent norm. Note also that $K^{0,0} = L^2$. A natural question is that of the relation of these spaces to ordinary Sobolev spaces $H^s(\mathbb{R}_+)$. If s is a non-negative integer, then the use of Hardy's inequality and iteration of the relation $\mathrm{d}/\mathrm{d}t = t^{-1} t\,\mathrm{d}/\mathrm{d}t$ gives that $K^{s,s} = H_0^s(\mathbb{R}_+)$ [3]. One can actually extend this result to all $s > -1/2$ but the proof is rather technical, and we refer the reader to [10]. To understand the relation to the full Sobolev space $H^s(\mathbb{R}_+)$ we have to add terms that correspond to non-vanishing traces. These terms will be of the form $t^k\omega(t)$ for integer k. Since singularities of more general form will also be needed to describe the asymptotic behaviour of solutions we will slightly increase the generality to allow for both. We start with the following simple lemma.

Lemma 1.3. *Let* $\omega_{p,k} = t^p \ln^k t\omega(t)$. *Then the Mellin transform* $\tilde{\omega}_{p,k}$ *is meromorphic in the plane; actually it is of the form* $(z-p)^{-k-1}h(z)$ *where* h *is entire and decreases rapidly along lines parallel to the imaginary axis.*

Proof. This is simple: it is easy to see that $(t\,\mathrm{d}/\mathrm{d}t)^{k+1}(t^{-p}\omega_{p,k})$ is a smooth function with compact support in $\mathbb{R}_+$. Hence the result follows from proposition 1.1 and the Paley–Wiener theorem. □

The essential information here is contained in the powers p and k, and this is encoded into the following definition.

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[2] It is easy to see using Paley–Wiener-type theorems that elements in $K^{s,\gamma}$ are analytic in the half-plane $\{z;\ \mathrm{Re}\, z > 1/2 - \gamma\}$ and increase there at most polynomially along lines parallel to the imaginary axis. Hence the spaces $K^{s,\gamma}$ decrease with increasing s and γ.

[3] Here $H_0^s(\mathbb{R}^+)$ is the closure of C_0^∞ in $H^s(\mathbb{R}_+)$. One can show that it is the space of those distributions in $H^s(\mathbb{R}_+)$ whose traces up to order (s) vanish at zero.

Definition 1.4. *A singularity type (at the origin)* in $\mathbb{R}_+$ is a countable set of pairs

$$\mathcal{P} = \{(p_j, m_j);\ p_j \in \mathbb{C}, m_j \text{ are non-negative integers, } j \in \mathbb{N}\}$$

such that $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$. To a given singularity type $\mathcal{P}$ and a fixed cut-off function ω we attach the set of *singular functions* given by

$$\mathrm{SF}(\mathcal{P}) = \{t^{-p_j} \ln^k t\omega(t);\ j \in \mathbb{N}, 0 \leqslant k \leqslant m_j\}.$$

Note that the choice of ω affects the set $\mathrm{SF}(\mathcal{P})$ up to elements in $C_0^\infty(\mathbb{R}_+)$.

The first thing to notice is that $\omega_{p,k} \in K^{s,\gamma}$ if and only if $\operatorname{Re} p < 1/2 - \gamma$. So, for the space $K^{s,\gamma}$ the essential part of the singular set is captured by the space[4]

$$S(\mathcal{P}, \gamma) = \operatorname{span}\{\omega_{p_j,k} \in \mathrm{SF}(\mathcal{P});\ \operatorname{Re} p_j > 1/2 - \gamma\},$$

and then we define

$$K^{s,\gamma}_{\mathcal{P}} = K^{s,\gamma} \oplus S(\mathcal{P}, \gamma).$$

Let us now go back to the relation between $K^{s,\gamma}$ and Sobolev spaces. The important singularity type here is the *Taylor singularity* given by

$$\mathcal{T} = \{(-j, 0);\ j \in \{0, 1, \ldots\}\}.$$

This corresponds precisely to adding terms with a given Taylor expansion at the origin. Then we have

$$H^s(\mathbb{R}_+) = K^{s,s} \oplus \mathcal{T}(s), \qquad s > 1/2, \quad s \neq 1/2 \bmod \mathbb{Z}$$

and

$$H^s(\mathbb{R}_+) = K^{s,s}, \qquad s < 1/2.$$

Note that at $s = 1/2$ this correspondence breaks down. These spaces can be defined on intervals $I \subset \bar{\mathbb{R}}_+$ simply by restricting to obtain spaces $K^{s,\gamma}_{\mathcal{P}}(I)$. If working on a graph, one usually has different singular behaviour at different end-points of the edges, and thus needs to employ two cut-offs for each edge localizing the behaviour to different end-points, and generally also two different weights and singularity types. Also, instead of asymptotics at zero, one may consider them at ∞ by making a change of variables $s = 1/t$.

For the Fredholm theory the essential property is the analogue of the Rellich lemma.

Proposition 1.5. *Let $I \subset \bar{\mathbb{R}}_+$ be a compact interval with zero as its left end-point. Then the imbedding*

$$K^{s,\gamma}(I) \to K^{s',\gamma'}(I), \qquad s \geqslant s', \quad \gamma \geqslant \gamma',$$

is compact if and only if $s > s'$ and $\gamma > \gamma'$.

By proposition 1.5 one also needs improvement in decay at the origin to obtain compactness. As we will see later, this fact implies that the double-layer potential operator, which is compact on a smooth bounded curve, fails to be compact on a boundary of a polygon.

Outline of the proof. Probably the easiest way to prove this is to use an alternate description of the space $K^{s,\gamma}$. Let $\epsilon : \mathbb{R} \to \mathbb{R}_+$ be a diffeomorphism such that $\epsilon(t) = \mathrm{e}^t$, $t < -1$, and $\epsilon(t) = t$, $t > 1$. Let g be a smooth positive function on $\mathbb{R}_+$ equal to t near zero, and equal to 1 outside a neighbourhood of zero. Then it is not hard to see that $u \in K^{s,\gamma}$ if and only the pull-back function (or distribution) $\epsilon^*(g^{-\gamma}u)$ belongs to the usual Sobolev space $H^s(\mathbb{R})$. The compact interval I changes under ϵ^{-1} to the half-open interval $J = (-\infty, a]$, and the improvement in s together with the usual Rellich lemma then shows that this imbedding is compact on any bounded subinterval of J, and the improvement in the weight γ brings in an exponentially decreasing weight factor that makes the imbedding also compact at $-\infty$. □

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[4] The span is here taken over $\mathbb{C}$, of course.

2. Mellin and Green operators

Consider the single-layer potential on $\mathbb{R}_+$ with a Newtonian kernel:

$$S\phi(t) = (2\pi)^{-1}\int_0^\infty \log(|s-t|)\phi(s)\,\mathrm{d}s. \tag{2.1}$$

We can alternatively write this as

$$S\phi(t) = \frac{t}{2\pi}\int_0^\infty \log(|1-t/s|)\frac{s}{t}\phi(s)\,\frac{\mathrm{d}s}{s} + K\phi(t)$$

where the operator $K\phi(t) = (2\pi)^{-1}\int_{\mathbb{R}_+}\phi(s)\log s\,\mathrm{d}s$ is one dimensional. Hence, modulo a one-dimensional operator, the single layer S is a convolution of $s^{-1}\ln|1-s|$ and ϕ with respect to the Haar measure of the multiplicative group $\mathbb{R}_+$ (which is just $\mathrm{d}s/s$), multiplied with $t/2\pi$. The Mellin transform, being the Fourier transform in the group $\mathbb{R}_+$, takes a convolution into a product of Mellin transforms, and hence, taking into account the factor of t in front causing a shift by 1, we have, for $\phi \in C_0^\infty(R_+)$,

$$M(S\phi)(z) = (2\pi)^{-1}M(t^{-1}\ln|1-t|)(z+1)\tilde{\phi}(z+1).$$

This is a typical example of a Mellin operator. A standard application of the residue theorem gives that

$$M(t^{-1}\ln|1-t|)(z) = \frac{c}{z-1}\frac{\cos(\pi(z-1))}{\sin(\pi(z-1))}.$$

This has all the typical features of a *Mellin symbol*, which we define precisely in a moment: first of all, it is a meromorphic function (a double pole at $z = 1$ and simple poles at all other integers); secondly, along lines parallel to the imaginary axis it has at most a polynomial growth (in this case it decays like $|z|^{-1}$). More generally we have the following definition:

Definition 2.1. A meromorphic function h is called a *Mellin symbol* if

(a) each strip $\{z;\ a \leqslant \operatorname{Re} z \leqslant b\}$ contains only finitely many poles of h; and
(b) there is an integer k such that for all real $a < b$ there is a constant $C = C_{a,b}$ such that

$$|h(z)| \leqslant C(1+|z|)^k, \qquad a \leqslant \operatorname{Re} z \leqslant b.$$

This is more general than the definition used in [11, section 1.3], and corresponds to the remark made after definition 1.3.5 there. On the other hand, this definition only allows for *constant coefficient symbols*, i.e. Mellin multipliers not depending on t. This is good enough for us since we will in this work only consider polygonal graphs. If the edges were arbitrary curves, one would need variable coefficient Mellin symbols, and the theory for these can be found in the second chapter of [11].

Next we show the quantization rule for Mellin calculus, i.e. how one associates an operator with a given symbol h. If h is a Mellin symbol, which for the moment we assume to have no poles on the line $\operatorname{Re} z = 1/2$, we define for $\phi \in C_0^\infty(R_+)$,

$$\mathrm{op_M}[h](\phi)(t) = M^{-1}(h\tilde{\phi})(t),$$

where M^{-1} is the inverse Mellin transform (with integration over the line $\operatorname{Re} z = 1/2$) defined in the previous section. This is still not quite what is meant by a Mellin operator; that is, one wants to localize it to zero, and also allow for some shifts and weights.

Definition 2.2. An operator of the form

$$\phi \mapsto t^{j-\gamma}\omega\,\mathrm{op_M}[\tau_\gamma h](t^\gamma\omega\phi),$$

with ω a cut-off at zero, h a Mellin symbol, and $j \in \mathbb{R}$, is called a *Mellin operator of conormal order* $-j$. Here τ_γ is the shift by γ, i.e. $\tau_\gamma f(z) = f(z+\gamma)$, and γ is some real weight factor.

So, going back to the single layer, we see that it determines a Mellin operator of conormal order -1. What about mapping properties of Mellin operators in $K^{s,\gamma}$ spaces? Let h now be a Mellin symbol, and let k be the smallest integer for which (b) in definition 2.1 holds for all $a < b$. If one wants to estimate the $K^{s',\gamma'}$ norm of $A\phi$ with A a Mellin operator with symbol h and conormal order $-j$, in terms of the $K^{s,\gamma}$ norm of an initially smooth ϕ, one observes two things: the Sobolev smoothness of $A\phi$ is affected only by the order of increase in $(1 + |z|)$, i.e. by k. Hence we may choose $s' = s - k$. On the other hand, the conormal order $-j$ will cause a shift by $-j$ on the line of integration, and if we want to move this back to line $\operatorname{Re} z = 1/2 - \gamma$ we may do so using the Cauchy theorem, *if there are no poles of h on the strip determined by real parts γ and $\gamma - j$*. Otherwise we will get additional terms with finite-degree poles on the Mellin side and rapid decrease along lines parallel to imaginary axis (since we assumed $\phi \in C_0^\infty(\mathbb{R}_+)$), and these correspond to singular terms of the type encountered earlier: $\omega(t)t^{-p}\ln^k t$. In conclusion, the above reasoning proves

Proposition 2.3. *Let h be a Mellin symbol satisfying (b) in definition 2.1, and let A be a Mellin operator with Mellin symbol h and conormal order $-j$. Then there is a continuous extension*

$$A : K^{s,\gamma} \to K^{s-k,\gamma-j} + S(\mathcal{P}, \gamma),$$

where the singularity type $\mathcal{P}$ is determined by the poles of the Mellin symbol.

Let us consider another example, even more relevant from the point of view of graphs. Consider the following (local) model. Let Γ_i, $i = 1, \ldots, k$, be half-lines in the plane with a common initial point at the origin. Let again S be the single layer on $\Gamma = \cup\Gamma_i$. This means that S is the operator given by the convolution with the function $(2\pi)^{-1}\ln|t|$. Here each edge Γ_i is equipped with its natural Lebesgue measure. Then any distribution $u \in \mathcal{E}'(\Gamma \setminus 0)$ is determined uniquely by the restrictions $u_i = u|_{\operatorname{int}\Gamma_i}$ and this gives a natural identification between $\mathcal{E}'(\Gamma \setminus 0)$ and $\oplus_i \mathcal{E}'(\operatorname{int}\Gamma_i)$. Similarly, we define $u \in K^{s,\underline{\gamma}}$, where $\underline{\gamma} = (\gamma_1, \ldots, \gamma_k)$ is a weight vector, if $u_{\Gamma_i} \in K^{s,\gamma_i}$. If all γ_i are equal to γ, then we simplify the notation to just $K^{s,\gamma}$. Under this decomposition the single layer S corresponds to the matrix operator (S_{ij}), where we define

$$S_{ij}\phi(t) = \int_{\Gamma_j} G(t - s)\phi(s)\,\mathrm{d}s, \qquad t \in \Gamma_i,$$

and we have used the notation $G(t) = (2\pi)^{-1}\ln|t|$. If α is the angle between Γ_i and Γ_j we may choose the coordinates above so that $t \in \mathbb{R}_+$ and the vectors on Γ_j are of the form $\mathrm{e}^{\mathrm{i}\alpha}s$, with s positive. This leads to an integral operator of the form

$$S_\alpha\phi(t) = \int_0^\infty G(t - \mathrm{e}^{\mathrm{i}\alpha}s)\phi(s)\,\mathrm{d}s, \qquad t > 0.$$

Again an easy computation shows that modulo a one-dimensional operator this is a Mellin convolution, multiplied with t, and the Mellin transform of the convolution kernel is

$$s_\alpha(z) = c(z-1)^{-1}\frac{\cos((\pi - \alpha)(z-1))}{\sin(z-1)}.$$

This is again meromorphic, with integer poles, but if $\alpha \in (0, 2\pi)$, this function is rapidly decreasing along lines parallel to the imaginary axis, and hence the operator is infinitely smoothing in the interior, as is obvious from the definition. However, the conormal order, which controls the weight at the origin, is still equal to -1! Hence this operator will not be compact as an operator of order -1 on $K^{s,\gamma}$ spaces[5]. This is a feature of *so-called Green operators*, which are an essential part of the calculus.

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[5] One can also think of this specific operator as *a smoothing Mellin operator of conormal order* -1.

The formal definition is slightly more complicated because we have additional natural operators that we also need to be included in this class. First of all, in the definition of a Mellin operator we did not state what happens if one changes the cut-off function, or, secondly, changes the γ to γ'. The last question is easy to answer; namely, when moving the integration one may cross some poles of h, and these will yield a finite-dimensional operator, whose image consists of distributions smooth in the interior, but singular at 0 (i.e. terms such as $\omega(t)t^p \ln^k t$). In the first case it is also easy to write down the kernel of the difference

$$\omega\, \mathrm{op}_{\mathrm{M}}[h]\omega - \omega'\, \mathrm{op}_{\mathrm{M}}[h]\omega'. \tag{2.2}$$

If we write $f = M^{-1}h$, then this kernel is just

$$\omega(t) f(t/t')\omega(t') - \omega'(t) f(t/t')\omega'(t').$$

Using a version of the Paley–Wiener theorem saying that if $u \in L^2(\mathbb{R}_+)$ with a compact support contained in $[a^{-1}, a]$ with $a > 1$, then the Mellin transform $\tilde{u}$ is analytic in the half-plane $\mathrm{Re}\, z < 1/2$ and decreases there rapidly along lines parallel to the imaginary axis, one sees that the kernel above lies in the tensor product $L^2(\mathbb{R}_+) \otimes_\pi K^{s,\gamma}_{\mathcal{P}}$ for some s and γ and a singularity type determined by h. For us the important operators will be the ones arising from interior smoothing Mellin operators[6], and then we can take s and γ to ∞. One can show that the spaces $K^{s,\gamma}$ are nuclear and hence an equivalent characterization for this property is that the operator in (2.2) maps boundedly $L^2(\mathbb{R}_+) \to K^{s,\gamma}_{\mathcal{P}}$. This is the formal definition of a Green operator; as already remarked, a Green operator might have a non-trivial conormal order, and hence it might affect the Fredholm properties of the problem.

The classes of Mellin and Green operators have the following composition properties (we assume from now on that kernels of Green operators are in $L^2(\mathbb{R}_+) \otimes_\pi K^{\infty,\infty}_{\mathcal{P}}$).

Proposition 2.4. *A product of a smoothing Mellin and a Green operator is a Green operator, and if A_1 and A_2 are two smoothing Mellin operators with Mellin symbols h_1 and h_2, then the product $A_1 A_2$ is a sum of an Mellin operator with a Mellin symbol h and a Green operator. Furthermore, if the operator A_i is of conormal order c_i, then $A_1 A_2$ is of conormal order $c_1 + c_2$.*

We will not prove this result, even though the proof is not difficult, and amounts to just carefully writing down the composition and using the definition of the Mellin and Green operators.

3. Pseudodifferential operators

We have localized the definition of the Mellin operator to a neighbourhood of the origin. Outside of this neighbourhood we wish to use the usual pseudodifferential definition of this operator. To accomplish this and still have a working operator (and a symbolic) calculus we need to understand what happens to a product of a pseudodifferential operator and a Mellin or a Green operator. So, let $a(\xi)$ be a (constant coefficient) classical pseudodifferential symbol on $\mathbb{R}_+$, and let

$$\mathrm{op}_\psi(a)\phi(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}x\xi} a(\xi)\hat{\phi}(\xi)\, \mathrm{d}\xi$$

be the corresponding pseudodifferential operator. The first thing that one can prove is the following result.

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[6] By this we mean the operators arising from Mellin symbols h that are rapidly increasing along lines parallel to the imaginary axis.

Lemma 3.1. *Let* ω_1 *and* ω_2 *be two cut-off functions with* $\omega_2 = 1$ *in the support of* ω_1*. Then the operators* $(1-\omega_2)\,\mathrm{op}_\psi(a)\omega_1$ *and* $\omega_1\,\mathrm{op}_\psi(a)(1-\omega_2)$ *are Green operators, with only Taylor asymptotics at the origin.*

Proof. This follows immediately from the pseudolocality of $\mathrm{op}_\psi(a)$ and of the definition of the Green operators. The only singularities possible are the Taylor asymptotics, since both of these operators map onto functions smoothly extendable to the whole of $\mathbb{R}$. □

Let us now consider the effect of restricting to $\mathbb{R}_+$ (or $\mathbb{R}_-$) on the symbolic level. Let

$$g_-(z) = (1 - \mathrm{e}^{2\pi\mathrm{i}z})^{-1}, \qquad g_+(z) = 1 - g_-(z).$$

Then one can show (see [10], p 56) that the Fourier multiplier $\theta_\pm$, with θ_+ the Heaviside function and $\theta_- = 1 - \theta_+$, determines a Mellin operator with Mellin multiplier $g_\pm$; i.e.,

$$F^{-1}(\theta_\pm\hat{\phi}) = M^{-1}(g_\pm\tilde{\phi})$$

for functions ϕ supported on the positive half-axis. Let us for the moment assume that the symbol a is of order zero, and classical; hence it admits a symbolic expansion

$$a \sim \sum_{j=0}^{\infty} a_{-j},$$

where a_{-j} is positively homogeneous of degree $-j$. Then we can write

$$a_{-j}(\xi) = a_{-j}^{+}|\xi|^{-j}\theta_+ + (-1)^j a_{-j}^{-}|\xi|^{-j}\theta_-,$$

and especially for the principal symbol a_0 we get

$$a_0(\xi) = a_0^+\theta_+ + a_0^-\theta_-.$$

An important class of operators is that for which $a_j^+ = a_j^-$. In this case we say that the symbol a (and also the operator $\mathrm{op}_\psi(a)$) has the *transmission property*[7]. Hence, for the principal part we have the representation

$$\mathrm{op}_\psi(a_0)\phi = \mathrm{op}_\mathrm{M}(\sigma_0),$$

where $\sigma_0(z) = a_0^+ g_+(z) + a_0^- g_-(z)$, and one can continue this also for the lower order terms. However, for us the important thing is the behaviour in the leading order, and with this in mind we formulate the following theorem.

Proposition 3.2. *If* a *is a constant coefficient classical symbol of order zero, then*

$$\omega\,\mathrm{op}_\psi(a)\omega = \omega\,\mathrm{op}_\mathrm{M}(\sigma_0)\omega + r_1,$$

with σ_0 *as above, and* r_1 *determines a Hilbert–Schmidt operator* $H^s(\mathbb{R}_+) \to H^s(\mathbb{R}_+)$ *(and hence a compact operator) for any real* s*.*

We do not prove this theorem, but the main idea has already been described above. The remainder is actually an operator of Green type, with a kernel in some singular spaces. Also, one can extend this representation in some sense to the complete symbol; i.e., there is a sequence of Mellin symbols of decreasing conormal orders $(\sigma_j(z))$ such that

$$\omega\,\mathrm{op}_\psi(a)\omega = \sum_{j=0}^{N} \omega\,\mathrm{op}_\mathrm{M}(\sigma_j)\omega + r_{N+1},$$

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[7] This is originally due to Boutet de Monvel [1], of course in a much more general setting, i.e. for smooth compact manifolds with a boundary. In [10] these results were extended to more general symbols. Also, for a very nice exposition on the transmission property, see [3].

where the kernel of r_{N+1} gets smoother and smoother as n increases. For the proof of this in the case of general parameter dependent symbols we refer the reader to [11, theorem 1.4.1.5]. Also, one can consider symbols of arbitrary order μ by taking out a factor $|\xi|^{-\mu}$, which corresponds to convolution with the kernel $c|x|^{-1-\mu}$, and this can be considered as a Mellin convolution of conormal order $-\mu$ in the way we already indicated.

4. Conormal symbols and ellipticity

Let us now describe the operator calculus in detail, and also indicate how one can define the notion of ellipticity so that elliptic operators correspond to operators having a parametrix in this class. The operator class will have two parameters: *the interior order* s and *conormal order* j. Operators of interior order s and conormal order j are of the form

$$r_+ \operatorname{op}_\psi(a) + \omega \operatorname{op}_{\mathrm{M}}(h)\omega + G, \tag{4.1}$$

where a is a classical constant coefficient symbol of order s, r_+ denotes the restriction to $\mathbb{R}_+$, h is a *smoothing* Mellin symbol of conormal order j, and G is a Green operator of lower conormal order than s or j. The reader now probably wonders why we consider only smoothing Mellin symbols. The reason is simple: in the examples that we have in mind, the only contribution into the operator which is not smoothing in the interior comes form a restriction of a classical pseudodifferential operator to $\mathbb{R}_+$, and on the other hand we already indicated that the operators on graphs corresponding to restriction to edges intersecting the support of the integrand only at a vertex, i.e. S_α for $\alpha \neq 0, 2\pi$, are also smoothing Mellin operators. Hence the structure of the algebra can be further restricted without violating its usefulness. As has been indicated, the above operators form an algebra which is bi-graded by the interior order and the conormal order. With an operator of type (4.1) we associate a *principal symbol* of order (s, j) as a pair $(\sigma_\psi(a), \sigma_{\mathrm{M}}(a_s) + h)$ where the first component $\sigma_\psi(a)$ is just the principal symbol of (order s) a, which is now called the *interior principal symbol*, and the second component consists of the Mellin symbol corresponding to $\omega \operatorname{op}_\psi(a_s)\omega$ (which is of conormal order s) and the Mellin symbol of $\operatorname{op}_{\mathrm{M}}(h)$ which is of conormal order j. The total conormal order is then $\max\{s, j\}$. In all the examples we will encounter, $j = s$, and then we call $\sigma_{\mathrm{M}}(a_s) + h$ *the principal conormal symbol of order* s.

We now introduce the following important definition:

Definition 4.1. Let the operator (4.1) be of order m, i.e. both its interior and conormal orders are equal to m. If γ is a real number (it corresponds to weight), we say that (4.1) is elliptic with respect to γ if

- the interior symbol is elliptic in the usual sense,
- the principal conormal symbol of order m has no zeros on the line $\operatorname{Re} z = 1/2 - \gamma$.

Elliptic symbols are precisely those that have inverses in the symbol class modulo symbols of lower order.

Theorem 4.2. *Let* $(\sigma_\psi(a), \sigma_{\mathrm{M}}(a_m) + h)$ *be an elliptic symbol of order* m. *Then there is an elliptic symbol of order* $-m$, $(\sigma_\psi(b), \sigma_{\mathrm{M}}(b_{-m}) + h')$, *such that* ab *is a classical symbol of order* -1 *and* $(\sigma_{\mathrm{M}}(a_m) + h)(\sigma_{\mathrm{M}}(b_{-m}) + h')$ *has conormal order* -1.

Again, we do not present a proof, even though it is not difficult. The definition of b and h' is rather obvious, and the only work is in the details when checking the correct symbolic behaviour.

Let us now consider a graph with edges Γ_i, $i \in I$, and edges v_k, $k \in K$. We assume, for simplicity[8], that all the vertices are bounded. Also we assume that edges intersect only at vertices and that at every vertex, only a finite number of edges intersect. Let Γ_i coincide with $[a_i, b_i]$ (i.e. a line segment connecting a_i and b_i) in a neighbourhood of its end-points. In general we have to allow different conormal orders for each end-point a_i and b_i, and a different interior order s_i for each edge. Again, to treat the classical layer potentials (and their parametrices) one only needs to consider the case when all of these are equal to m, and this we will do. Now, for a vertex e_k let $I(k)$ be the set of those i such that e_k is an end-point of Γ_i, and for an $r \in I(k)$ let ω_r^k be a function in $C^\infty(\Gamma_r)$ such that it vanishes outside a neighbourhood of e_k and is equal to one near a smaller neighbourhood. On the graph our operators of order m will be of the form

$$A = \sum_i r_i \operatorname{op}_\psi(a^i) + \sum_k \sum_{r,s\in I(k)} \omega_r^k \operatorname{op}_{\mathrm{M}}(h_{rs}^k)\omega_s^k + \sum_i G_i,$$

where r_i is the restriction to Γ_i, a_i are classical symbols of order m which are independent of t near the end-points, h_{rs}^k are Mellin symbols of conormal order m, and finally G_i are smoothing Green operators of lower conormal order[9]. The principal symbol of A is defined as follows: the principal interior symbol is the family $(\sigma_\psi(a_i))_{i\in I}$. The principal conormal symbol is parametrized by the set of vertices, and it consists of matrix valued meromorphic functions h_k, where the diagonal elements of h_k are of the form

$$\operatorname{op}_{\mathrm{M}}(a_m^i)(z) + h_{ii}^k(z), \qquad i \in I(k),$$

and the off-diagonal terms are of the form

$$h_{rs}^k(z), \qquad r, s \in I(k), \quad r \neq s.$$

The ellipticity of an operator on a graph is defined then in the obvious manner: all the interior symbols $\sigma_\psi(a^i)$ have to be elliptic, and all the principal conormal symbols associated with vertices have to be invertible matrices on the line $\operatorname{Re} z = 1/2 - \gamma$. In view of the invertibility result of theorem 4.2 and the generalization of the Rellich lemma, we get the following result.

Theorem 4.3. *If A is an elliptic operator of order m on a compact graph with respect to a weight factor γ, then there is a singularity type*[10] *$\mathcal{P}$ such that the operator*

$$A : K^{s,\gamma} \to K_{\mathcal{P}}^{s-m,\gamma-m}$$

defines a Fredholm map.

As already mentioned, the assumption that the graph is compact is not that essential. One only has to associate a weight factor and a conormal symbol for each unbounded end of the graph.

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[8] Treating the unbounded ends of vertices does not cause any major difficulties; one just localizes the behaviour there and uses the modified version of the calculus described here.

[9] The operator $\omega_r^k \operatorname{op}_{\mathrm{M}h}\omega_s^k$ is defined as follows: one identifies both Γ_r and Γ_s with closed line segments of $\mathbb{R}_+$ starting at zero, and then for a function supported in Γ_s localizes it to a neighbourhood of zero, multiplies the Mellin transform by h, and takes as the value a function supported on Γ_r localized to a neighbourhood of e_k.

[10] The singularity type is defined as a collection of singularity types, one for each vertex, and the spaces are then defined in the obvious manner by localizing them to different vertices. Also, the singularity type is determined by the poles of the conormal symbols.

5. Applications to graphs

Let now $\Gamma = \cup\Gamma_i$ be a compact graph, with edges Γ_i and vertices e_k, and denote by Ω_l the set of *cells* determined by Γ, i.e. the connected components of $\mathbb{R}^2 \setminus \Gamma$. We continue to assume that the graph satisfies the assumptions made in the previous section. In particular there are no unbounded ends; hence there is only one unbounded component. We also assume that at every vertex, at least two edges intersect; hence there are no slit parts of the boundary, i.e. no interior angles of 2π.

Let us now briefly recall the definition of the Dirichlet to Neumann map as described in [7]. We look for a family of functions u_l harmonic in the cells Ω_l and such that the function u_0 associated with the unbounded component, which we denote by Ω_0, satisfies the radiation condition

$$u_0(z) = A \ln|z| + B + \mathrm{o}(1) \qquad \text{as } |z| \to \infty$$

for some constants A and B. We want the function determined by the set $\{u_l\}$ to be extendable as continuous functions across Γ, and hence one may use an ansatz as a single-layer potential

$$u_l = S(f)|_{\Omega_l}, \tag{5.1}$$

where

$$Sf(x) = (2\pi)^{-1} \int_\Gamma \ln|x-y| f(y)\,\mathrm{d}S(y).$$

Then, assuming $1 > s > -1$ and $f \in H^s(\Gamma)$, the restrictions u_l will be in $H^{s+3/2}$ up to Γ in each cell[11]. Now let g be a continuous function on Γ. If (u_l) is a collection of harmonic functions in the cells as described above such that on the Γ it has trace equal to g, one may look at the jump of the normal derivatives across Γ; i.e. if Ω_r and Ω_s are two cells, whose boundaries intersect along Γ_i, let n_{rs} be the unit normal to Γ_i pointing from Ω_r to Ω_s. Let

$$[\partial u/\partial n]_{rs} = \partial u_r/\partial n_{rs}|_{\Gamma_i} - \partial u_s/\partial n_{rs}|_{\Gamma_i},$$

where the traces are to be understood in the weak sense via Green formulae. Note that changing the order of r and s will not change the jump! The Dirichlet to Neumann map Λ_Γ is defined as follows: let g be the trace of u on Γ; then

$$\Lambda_\Gamma : g \mapsto ([\partial u/\partial n]_{rs})$$

where r and s run over all pairs with a common intersection along some Γ_i.

If u is given as a single-layer potential $S(f)$, then by the jump relations for the normal derivative of a single-layer potential we get

$$\Lambda_\Gamma(g) = f.$$

On the other hand, since $g = S(f)$ on the graph, we get that

$$S\Lambda_\Gamma = \mathrm{Id},$$

and if S is invertible[12], one also gets

$$\Lambda_\Gamma S = \mathrm{Id},$$

and hence then $\Lambda_\Gamma = S^{-1}$.

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[11] This follows by analysing the boundary behaviour of Sf if Ω_l using the Mellin transform in the conormal direction; for the details, refer to [10].

[12] This means that actually the Dirichlet problem on the graph is uniquely solvable, and hence equivalent to the existence of the Dirichlet to Neumann map. One could also relax this assumption by defining the Dirichlet to Neumann map only as the Cauchy data of all harmonic functions representable as single-layer potentials. Then the above relation is always true.

To this end let us consider an example: assume Γ is such that all intersections are symmetric triple junctions. Then the interior symbol of S on each Γ_i is just $|\xi^{-1}|$, and hence the interior ellipticity holds trivially. At a vertex the local model for the single layer is the matrix operator

$$\begin{pmatrix} S_0 & S_\alpha & S_\alpha \\ S_\alpha & S_0 & S_\alpha \\ S_\alpha & S_\alpha & S_0 \end{pmatrix}$$

where $\alpha = 2\pi/3$ and S_α and S_0 are defined as before. Here, S_0 corresponds to the pseudodifferential operator $\mathrm{op}_\psi(1/|\xi|)$ on each edge, and S_α are the (smoothing) Mellin operators. To compute the zeros and poles of the principal conormal symbol, it is useful to perform the following transformation which diagonalizes the above matrix. If $u_i = u|\Gamma_i$, $i = 1, \ldots, 3$, then let

$$v_1 = u_1 + u_2 + u_3, \qquad v_2 = u_1 - u_2, \qquad v_3 = u_1 - u_3.$$

Then for v, the single layer acts as a diagonal matrix

$$3\begin{pmatrix} S_0 + 2S_{2\pi/3} & 0 & 0 \\ 0 & S_0 - S_{2\pi/3} & 0 \\ 0 & 0 & S_0 - S_{2\pi/3} \end{pmatrix}$$

and this gives the conormal symbol of order -1 [13]

$$\frac{c}{z\sin(\pi z)}\begin{pmatrix} \cos(\pi z) + \cos(\pi z/3) & 0 & 0 \\ 0 & \cos(\pi z) - \cos(\pi z/3) & 0 \\ 0 & 0 & \cos(\pi z) - \cos(\pi z/3) \end{pmatrix}.$$

The poles are all at integer points, and are all single except the one at zero, which is a double pole. This is because the kernel of the single layer will give a logarithmic contribution at the origin. The poles are also not hard to find: the zeros of $\cos(\pi z) + 2\cos(\pi z/3)$ are given by

$$z = 1, 2 + 3\mathbb{Z} \qquad \text{or} \qquad z = 3/2 + 3\mathbb{Z}.$$

Note that the integer zeros will get killed by the poles. Similarly, the zeros of $\cos(\pi z) - \cos(\pi z/3)$ are

$$z = 3\mathbb{Z} \qquad \text{or} \qquad z = 3/4 + 3\mathbb{Z}/2.$$

Using these observations we can easily determine when S on Γ is invertible.

Proposition 5.1. *Let Γ be a compact graph with only triple symmetric junctions. Assume that the exterior Dirichlet problem*

$$\Delta v = 0 \qquad \textit{in } \Omega_0, \quad v|\partial_{\Omega_0} = 0,$$

with at most a logarithmic growth at infinity, has only the trivial solution. Then

$$S : H^s(\Gamma) \to H^{s+1}(\Gamma)$$

is well defined and invertible when $-1 < s < -1/2$.

Proof. First of all, using the relation between $K^{s,\gamma}$ spaces and usual Sobolev spaces, one gets, in view of the location of poles and zeros of the conormal symbols of S, that

$$S : H_0^s(\Gamma) \to H^{s+1}(\Gamma) + L$$

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[13] We have now included the shift by 1 corresponding to t, so, as a Mellin multiplier, this acts on Mellin transforms of v_j at points $z + 1$.

is bounded for $s < 1/2$, where L is a finite-dimensional space that is spanned by $\omega_k \log|t|$ with ω_k a cut-off at vertex e_k. Now for $s < 1/2$, the space $C_0^\infty(\Gamma \setminus \cup\{e_k\})$ is dense in H^s, and since also $\omega_k \log|t|$ then belongs to $H^s(\Gamma)$, we get that

$$S : H^s(\Gamma) \to H^{s+1}(\Gamma), \qquad s < -1/2, \tag{5.2}$$

is bounded. Let us then consider the injectivity. Assume that $Sf = 0$, $f \in H^s$ with $s > -1$. Then if one defines $u = Sf$ in $\mathbb{R}^2 \setminus \cup\Gamma_i$, one has a harmonic function with zero boundary values on Γ; note that since $s > -1$, this u will be in $H^{s+3/2}(\mathbb{R}^2 \setminus \cup\Gamma_i)$ locally, and hence the traces exist. Thus, by the maximum principle and the uniqueness of the exterior Dirichlet problem, $u = 0$. Then the use of standard jump relations for the normal derivative of the single layer gives $f = 0$. Hence, for $-1 < s < -1/2$, the single layer $S : H^s \to H^{s+1}$ is injective. The surjectivity follows by considering the adjoint. Since the single layer is a symmetric operator, by taking adjoints we get the bounded extension

$$S : H^s(\Gamma) \to H^{s+1}(\Gamma), \qquad 0 > s > -1/2,$$

and the same reasoning as above shows that this is injective, and thus shows the surjectivity of (5.2) for $-1 < s < -1/2$. □

Thus S behaves on this smoothness range as a classical pseudodifferential operator of order -1, and its inverse Λ_Γ as a classical pseudodifferential operator of order 1. Now this implies also estimates for the spectrum of S (and hence of Λ_Γ too). That is, using the arguments from [12][14], one sees from the fact that the entropy numbers[15] ρ_k of the compact imbedding $H^{s+1}(\Gamma) \to H^s(\Gamma)$ behave as $\rho_k \sim k^{-1}$, and Carl's inequality connecting the eigenvalues to entropy numbers, that one has the same asymptotic behaviour for the eigenvalues of S.

Constant coefficient differential operators are represented on the Mellin side by diagonal matrices with polynomial entries. For example the Laplace–Beltrami operator has the Mellin symbol $z(z-1)$. One of the most interesting questions is how close the square of λ is to a second-order differential operator. Our calculus shows that there exists such a differential operator $P(D)$ that the difference of Λ_Γ^2 and $P(D)$ is an operator of lower order both in the Sobolev and in the conormal sense. Outside of the edges, the operator $P(D)$ is the Laplace–Beltrami operator. The drawback of the method is that the coefficients of the Taylor asymptotics cannot be obtained by this calculus since they are affected by the smoothing Green operators. We end this work with a comment on the general, non-symmetric case. One way to obtain estimates (even though crude) for Λ_Γ is to study the single layer on symmetric graphs, and its mapping properties. Generally, one cannot always hope to have behaviour like that above. See for example the case of a symmetric quadruple junction considered in [7]. The location of poles and zeros of the conormal symbols will affect the smoothing properties of S, and consequently the estimates can get worse. This effect can (at least in principle) be controlled by the conormal symbols. We plan to return to these matters in another work.

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[14] See especially the proof of theorem 25.2.

[15] This means that there are constants c_1 and c_2 such that $c_1 k^{-1} \leqslant \rho_k \leqslant c_2 k^{-1}$ for all k.

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