

Inverse scattering problem for two-dimensional Schrödinger operator

V. SEROV* L. PÄIVÄRINTA†

Received December 6, 2005

Abstract — This work deals with the inverse scattering problem for two-dimensional Schrödinger operator. The following problem is studied: To estimate more accurately first nonlinear term from the Born series which corresponds to the scattering data with all energies and all angles in the scattering amplitude. This estimate allows us to conclude that the singularities and the jumps of the unknown potential can be obtained exactly by the Born approximation. Especially, for the potentials from L^p -spaces the approximation agrees with the true potential up to the continuous function.

1. INTRODUCTION

The purpose of this work is to correct (to make more accurate) the estimates of the first nonlinear term from the Born approximation in the Schrödinger operator which was published in the works [6–8] and to formulate more precisely the results concerning the reconstruction of singularities of the unknown potential. The main result of the above-mentioned articles is that the leading order singularities (in some particular cases all singularities and jumps) of the potential are obtained exactly from the scattering amplitude by the linearized method (Born approximation). Actually in present article we will obtain more stronger results than in [6–8] for singular potentials in two-dimensional case. The notation “singular potential” in this article means that a potential is not locally bounded.

Let q be a real valued potential in \mathbb{R}^2 appearing in the Schrödinger operator

$$H = -\Delta + q(x).$$

*Department of Mathematical Sciences, University of Oulu, P.O. Box 3000, FIN-90014, Oulu, Finland. E-mail: vserov@cc.oulu.fi

†Department of Mathematics, University of Helsinki, P.O. Box 4, FIN-00014, Helsinki, Finland. E-mail: lassi.paivarinta@rni.helsinki.fi

We assume that the potential belongs to the weighted space $L^p_\sigma(\mathbb{R}^2)$ defined by the norm

$$\|q\|_{p,\sigma} = \left(\int_{\mathbb{R}^2} (1 + |x|)^{p\sigma} |q(x)|^p dx \right)^{1/p}, \quad (1.1)$$

where $1 < p \leq \infty$ and σ is a nonnegative number that will be specified later.

Below we use the following notations. The space $W^t_p(\mathbb{R}^2)$ denotes the usual L^p -based Sobolev space in \mathbb{R}^2 and $H^t(\mathbb{R}^2) = W^{t,2}_2(\mathbb{R}^2)$.

Under the above assumptions on the potential the Hamiltonian H is a self-adjoint operator in $L^2(\mathbb{R}^2)$. The spectrum of this operator consists of a continuous spectrum, filling out the positive real axis (with possible positive eigenvalues), and a possible negative discrete spectrum of finite multiplicity with zero as the only possible accumulation point. In addition we suppose the potential has some power decay at the infinity

$$|q(x)| \leq C|x|^{-\mu} \quad (1.2)$$

for large $|x|$ and for some $\mu > 2$. Then the discrete spectrum (if it exists) is purely negative and finite, there are no positive eigenvalues and zero belongs to the continuous spectrum $[0, \infty)$ (see [2, 11]). In this case we can define for arbitrary $k \in \mathbb{R}, k \neq 0$, the scattering solutions of the homogeneous Schrödinger equation

$$(H - k^2)u(x, k) = 0$$

to be the unique solutions of the Lippmann–Schwinger equation

$$u(x, k, \vartheta) = e^{ik(x,\vartheta)} - \int_{\mathbb{R}^2} G_k^+(|x-y|)q(y)u(y, k, \vartheta) dy,$$

where $\vartheta \in S^1$ and the outgoing fundamental solution of the corresponding Helmholtz equation G_k^+ is defined as

$$G_k^+(|x|) = \frac{i}{4} H_0^{(1)}(|k||x|),$$

where $H_0^{(1)}$ is the Hankel function of the first kind and 0 order. Recall that the function $G_k^+(|x-y|)$ is the kernel of the integral operator $(-\Delta - k^2 - i0)^{-1}$.

The solutions $u(x, k, \vartheta)$ for $k > 0$ admit asymptotically, as $|x| \rightarrow +\infty$ uniformly with respect to $\vartheta \in S^1$, a representation

$$u(x, k, \vartheta) = e^{ik(x,\vartheta)} + \frac{1+i}{4\sqrt{\pi}} e^{ik|x|} k^{-1/2} |x|^{-1/2} A(k, \vartheta', \vartheta) + o\left(\frac{1}{|x|^{1/2}}\right),$$

where $\vartheta' = x/|x| \in S^1$ and the function $A(k, \vartheta', \vartheta)$ is called a scattering amplitude and is defined as

$$A(k, \vartheta', \vartheta) = \int_{\mathbb{R}^2} e^{-ik(\vartheta', y)} q(y)u(y, k, \vartheta) dy. \quad (1.3)$$

For reasons of purely technical nature, we define the solutions $u(x, k, \vartheta)$ for negative values of k as

$$u(x, k, \vartheta) = \overline{u(x, -k, \vartheta)}. \quad (1.4)$$

Therefore, we can extend A to negative k by $A(k, \vartheta', \vartheta) = \overline{A(-k, \vartheta', \vartheta)}$ to obtain a well-defined scattering amplitude for all $k \in \mathbb{R}, k \neq 0$. Such extension of A exactly corresponds to (1.4).

Definition 1. We say that the Hamiltonian H has a resonance at zero if the homogeneous Lippmann–Schwinger equation for $k = 0$

$$v(x) = - \int_{\mathbb{R}^2} G_0^+(|x - y|)q(y)v(y) dy,$$

where $G_0^+(|x|) = -(2\pi)^{-1} \log |x| + c_0$ and c_0 is known positive constant, has a nontrivial continuous solution vanishing uniformly at the infinity.

Note that if $q(x) \geq 0$ or $\int_{\mathbb{R}^2} |G_0^+(|x - y|)| |q(y)| dy < 1$ then H has no resonance at zero. As it follows from [12] and [13] if the Hamiltonian has no resonance at zero then the scattering amplitude can be also well-defined for $k = 0$ by continuity and it is equal to zero in this case. The inverse scattering problem that is considered here is: to recover the potential (or its points of singularity) from the knowledge of $A(k, \vartheta', \vartheta)$ for all $k > 0, \vartheta'$ and ϑ .

It follows from (1.3) that for every fixed point $\xi \in \mathbb{R}^2$

$$(Fq)(\xi) = \lim_{k \rightarrow +\infty} A(k, \vartheta', \vartheta), \quad \xi = k(\vartheta - \vartheta'),$$

where F is the ordinary Fourier transform in \mathbb{R}^2 . If we write $\xi = k(\vartheta - \vartheta')$ then k and ϑ' can be obtained back as

$$k = \frac{|\xi|}{2(\vartheta, \hat{\xi})}, \quad \vartheta' = \vartheta - 2(\vartheta, \hat{\xi})\hat{\xi}, \quad \hat{\xi} = \frac{\xi}{|\xi|}. \tag{1.5}$$

The latter formulas justify the following definition.

Definition 2. The inverse Born approximation $q_B(x)$ of the potential $q(x)$ is defined as follows:

$$q_B(x) := \frac{1}{32\pi^3} \int_{\mathbb{R} \times S^1 \times S^1} e^{-ik(\vartheta - \vartheta', x)} A(k, \vartheta', \vartheta) |k| |\vartheta - \vartheta'|^2 dk d\vartheta d\vartheta'. \tag{1.6}$$

Note that the latter integral is just the inverse Fourier transform of A on the manifold $\mathbb{R} \times S^1 \times S^1$ (see [5, 6]).

It is very easy to see that within the Born approximation, the scattering amplitude is simply the Fourier transform of the unknown potential. The weaker the potential, the better this approximation. But even when the potential is not weak the Fourier transform of a scattering amplitude contains essential information of the potential as it was shown in [4, 6, 10, 14] in two dimensions. In this paper we improve these results for potentials with stronger singularities.

The following estimates for the resolvent of the Laplacian (see [9, 14]) on the continuous spectrum play the key role in this work.

Proposition 1. Assume that $1 < p \leq \infty$. Then for all $k \in \mathbb{R}, k \neq 0$, the limit

$$(-\Delta - k^2 - i0)^{-1} := \lim_{\epsilon \rightarrow +0} (-\Delta - k^2 - i\epsilon)^{-1}$$

exists in the uniform operator topology from $L_{\delta}^{2p/(p+1)}(\mathbb{R}^2)$ to $L_{-\delta}^{2p/(p-1)}(\mathbb{R}^2)$ with the norm estimate

$$\|(-\Delta - k^2 - i0)^{-1} f\|_{L_{-\delta}^{2p/(p+1)}(\mathbb{R}^2)} \leq \frac{C}{|k|^{\gamma}} \|f\|_{L_{\delta}^{2p/(p-1)}(\mathbb{R}^2)}, \tag{1.7}$$

where $\gamma = 2 - 2/p$ and $\delta = 0$ for $1 < p \leq 3/2$ and $\gamma = 1 - 1/2p$ and $\delta > 1/2 - 3/4p$ for $3/2 < p \leq \infty$.

The following corollary is more important for our purposes. Let us denote by \hat{K} the integral operator having the kernel

$$K(x, y) = |q(x)|^{1/2} G_k^+(|x - y|) q_{1/2}(y),$$

where $G_k^+(|x - y|)$ is the kernel of the integral operator $(-\Delta - k^2 - i0)^{-1}$ and $q_{1/2} = |q|^{1/2} \text{sign}(q)$.

Corollary 1. *Assume that the potential $q(x)$ belongs to $L_{2\delta}^p(\mathbb{R}^2)$ with $1 < p \leq \infty$ and with δ as above. Then the operator \hat{K} is bounded in $L^2(\mathbb{R}^2)$ with the norm estimate*

$$\|\hat{K}\|_{L^2 \rightarrow L^2} \leq \frac{C}{|k|^{\gamma}}, \tag{1.8}$$

where γ is as in (1.7).

Now let $\Phi_0(k)$ and $\Phi(k)$ be the operators, defined for $f \in L^2(S^1)$ as

$$\begin{aligned} \Phi_0(k)f(x) &= |q(x)|^{1/2} \int_{S^1} e^{ik(\vartheta, x)} f(\vartheta) d\vartheta, \\ \Phi(k)f(x) &= |q(x)|^{1/2} \int_{S^1} u(x, k, \vartheta) f(\vartheta) d\vartheta. \end{aligned} \tag{1.9}$$

Corollary 2 (see [6]). *Under the same assumptions for $q(x)$ as in Corollary 1, the operators $\Phi_0(k)$ and $\Phi(k)$ are bounded from $L^2(S^1)$ to $L^2(\mathbb{R}^2)$ with the norm estimates*

$$\|\Phi_0(k)\| \leq \frac{C}{|k|^{\gamma/2}}, \quad \|\Phi(k)\| \leq \frac{C}{|k|^{\gamma/2}}, \tag{1.10}$$

where γ is as in (1.7).

We are now in the position to represent the asymptotic expansion for the Born potential. A repeated use of the Lippmann–Schwinger equation yields the following representation for the Born potential $q_B(x)$:

$$q_B(x) - q(x) = q_1(x) + \sum_{j=2}^m q_j(x) + \tilde{q}_{m+1}(x), \tag{1.11}$$

where $q_j(x)$ and $\tilde{q}_j(x)$ have the forms

$$\begin{aligned} q_j(x) &= F_M^{-1}(\Phi_0^*(k) \text{sign}(q) \hat{K}^j \Phi_0(k)), \\ \tilde{q}_j(x) &= F_M^{-1}(\Phi_0^*(k) \text{sign}(q) \hat{K}^j \Phi(k)), \end{aligned} \tag{1.12}$$

where the inverse Fourier transform is applied to the kernel of the corresponding integral operator and $\Phi_0^*(k)$ is the adjoint operator for $\Phi_0(k)$. And for the first nonlinear term $q_1(x)$ we have a special representation (see (1.6)):

$$q_1(x) = \frac{1}{16\pi^3} \int_{\mathbb{R}^2 \times \mathbb{R}^2} q(y)q(z) dy dz \int_{-\infty}^{\infty} |k| \tilde{G}_k^+(|y-z|) dk \\ \times \int_{S^1 \times S^1} (1 - (\vartheta, \vartheta')) e^{-ik(\vartheta, x-z) - ik(\vartheta', y-x)} d\vartheta d\vartheta', \quad (1.13)$$

where $\tilde{G}_k^+ = G_k^+$ for $k > 0$ and $\tilde{G}_k^+ = \overline{G_k^+}$ for $k < 0$ (see (1.4)).

To estimate the smoothness of q_j we will prove the following lemmas.

Lemma 1.1. *Assume that the potential $q(x)$ satisfies all conditions of Corollary 1 with $3/2 < p \leq \infty$. Then the terms $q_j(x)$ and $\tilde{q}_j(x)$ from the Born expansion (1.11) for $j \geq 2$ belong to the Lipschitz class $Lip(\alpha)$ for any $\alpha \leq 1 - 3/(2p)$.*

Proof. For x_1, x_2 in \mathbb{R}^2 we have (see [5] and (1.6))

$$q_j(x_1) - q_j(x_2) = C \int_{-\infty}^{\infty} |k| dk \int_{S^1} d\vartheta \int_{S^1} d\vartheta' (1 - (\vartheta, \vartheta')) \\ \times \Phi_0^* \text{sign}(q) \hat{K}^j \Phi_0(k, \vartheta, \vartheta') (e^{-ik(\vartheta - \vartheta', x_1)} - e^{-ik(\vartheta - \vartheta', x_2)}).$$

If for $l = 1, 2$ we denote by $e_l = e^{ik(\vartheta, x_l)} \in L^2(S^1)$ and $E_l = \vartheta \cdot e_l \in (L^2(S^1))^2$ in the space of vector-valued L^2 -functions, then the latter difference will be equal to

$$C \int_{-\infty}^{\infty} |k| dk ((e_1, \Phi_0^* \text{sign}(q) \hat{K}^j \Phi_0 e_1)_{L^2(S^1)} - (e_2, \Phi_0^* \text{sign}(q) \hat{K}^j \Phi_0 e_2)_{L^2(S^1)} \\ - (E_1, \Phi_0^* \text{sign}(q) K^j \Phi_0 E_1)_{L^2(S^1)} + (E_2, \Phi_0^* \text{sign}(q) K^j \Phi_0 E_2)_{L^2(S^1)}). \quad (1.14)$$

Because $\|E_1 - E_2\|_{L^2(S^1)} = \|e_1 - e_2\|_{L^2(S^1)}$ and $\|e_l\|_{L^2(S^1)}^2 = |S^1| = 2\pi$, we obtain from (1.14)

$$|q_j(x_1) - q_j(x_2)| \leq C \int_{-\infty}^{\infty} |k| dk \|e_1 - e_2\|_{L^2(S^1)} \|\Phi_0^* \text{sign}(q) \hat{K}^j \Phi_0\|. \quad (1.15)$$

Next, let us stress now that (see, for example, [16])

$$\|e_1 - e_2\|_{L^2(S^1)}^2 = \int_{S^1} (2 - e^{ik(\vartheta, x_2 - x_1)} - e^{ik(\vartheta, x_1 - x_2)}) d\vartheta \\ = 4\pi(1 - J_0(|k||x_1 - x_2|))$$

where J_0 is the Bessel function of order zero. According to Corollary 1 and Corollary 2 and due to the fact that H has no resonance at zero we have

$$\|\Phi_0^* \text{sign}(q) \hat{K}^j \Phi_0\| \leq \frac{C}{1 + |k|^{\gamma(j+1)}}$$

with γ as in Proposition 1.

These two facts and (1.14) imply that (by denoting $r = |x_1 - x_2|$) we need an estimate for the integral

$$\int_0^\infty \frac{k}{1 + k^{\gamma(j+1)}} (1 - J_0(kr))^{1/2} dk. \tag{1.16}$$

We split this integral into two parts: $1/r < k < \infty$ and $0 < k < 1/r$. Using the asymptotic behavior of the Bessel functions for large argument we estimate the first part by

$$C \int_{1/r}^\infty \frac{k}{1 + k^{\gamma(j+1)}} dk \leq Cr^{\gamma(j+1)-2}$$

with the condition $\gamma(j + 1) > 2$. In order to estimate the remaining part of the integral (1.16) we use the following asymptotic behavior of the Bessel functions for small argument

$$J_0(x) = 1 + O(x^2),$$

If we take it into account then we can estimate the second part of (1.16) by

$$Cr \int_0^{1/r} \frac{k}{1 + k^{\gamma(j+1)}} dk \leq Cr \left(1 + \int_1^{1/r} \frac{1}{k^{\gamma(j+1)-2}} dk \right) \leq Cr^{\min(1, \gamma(j+1)-2)}.$$

Hence, the proof for $q_j(x)$ is complete if we can show that $q_j \in L^\infty$. But this is true, since

$$|q_j(x)| \leq C \int_0^\infty \frac{k}{1 + k^{\gamma(j+1)}} (\|e\|_{L^2(S^1)} + \|E\|_{L^2(S^1)}) dk \leq C < \infty$$

for $\gamma(j + 1) > 2$. It remains to observe that for $\gamma = 1 - 1/(2p)$ and $j \geq 2$ the condition $\gamma(j + 1) > 2$ implies $3/2 < p \leq \infty$. We leave it for the readers to check that the proof goes through for $\tilde{q}_j(x)$ with obvious changes. Lemma 1.1 is thus proved. \square

Let us prove a little bit more “rough” result for q_j but for any $j \geq 1$.

Lemma 1.2. *Assume that the potential $q(x)$ satisfies all conditions of Corollary 1 and in addition $q(x)$ belongs to $L^1(\mathbb{R}^2)$. Then for $j \geq 1$ the terms $q_j(x)$ and $\tilde{q}_j(x)$ belong to the Sobolev space $H^t(\mathbb{R}^2)$ for any $t < \gamma(j + 1/2) - 1$ and with γ as in Proposition 1.*

Proof. As it follows from (1.12) and (1.10) it is enough to prove the statement of this lemma only for $q_j(x)$. By the definition of the norm in the Sobolev space H^t and (1.12) we obtain

$$\begin{aligned} \|q_j\|_{H^t(\mathbb{R}^2)}^2 &= \|(1 + |\xi|^2)^{t/2} F(q_j)(\xi)\|_{L^2(\mathbb{R}^2)}^2 \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} (1 + |\xi|^2)^t d\xi \left| \int_{S^1} [\Phi_0^* \text{sign}(q) \hat{K}^j \Phi_0] \left(\frac{|\xi|}{2(\vartheta, \hat{\xi})}, \vartheta - 2(\vartheta, \hat{\xi})\hat{\xi}, \vartheta \right) d\vartheta \right|^2, \end{aligned}$$

where $[\Phi_0^* \text{sign}(q) \hat{K}^j \Phi_0]$ denotes the kernel of the corresponding operator inside. After the change of variables (1.5) we can get from the latter equality

$$\begin{aligned} \|q_j\|_{H^t(\mathbb{R}^2)}^2 &\leq C \int_{\mathbb{R} \times S^1 \times S^1} (1 + |k(\vartheta - \vartheta')|^2)^t |[\Phi_0^* \text{sign}(q) \hat{K}^j \Phi_0](k, \vartheta', \vartheta)|^2 \\ &\quad \times |k| |\vartheta - \vartheta'|^2 dk d\vartheta' d\vartheta \\ &\leq C \int_0^\infty k(1 + k^2)^t dk \int_{S^1} \int_{S^1} |\Phi_0^*(\text{sign}(q) \hat{K}^j (|q|^{1/2} e^{ik(\vartheta', \cdot)})|^2 d\vartheta d\vartheta' \\ &\leq C \int_0^\infty k(1 + k^2)^t dk \int_{S^1} \|\Phi_0^*\|^2 \|\hat{K}\|^{2j} \|q\|_{L^1} d\vartheta \\ &\leq C \left(1 + \int_1^\infty \frac{k^{1+2t} dk}{k^{\gamma(2j+1)}}\right). \end{aligned}$$

But the latter integral converges if and only if $t < \gamma(j + 1/2) - 1$. Therefore, Lemma 1.2 is proved. \square

2. ANALYSIS OF THE FIRST NONLINEAR TERM AND MAIN THEOREM

In this section it will be shown that the smoothness estimates obtained in the previous section are not the best possible, especially for $q_1(x)$ (see Lemma 1.2). That's why we will try to write the first nonlinear term in a more explicit form to refine the analysis of the degree of smoothness.

Let us remark also that the final step (the inequality (4.14)) in the proof of Lemma 2.4 in [7] is not correct. The same situation with the proof of Theorem 2.2 in [6]. And in this section we will give the revised version of the corresponding proofs in two dimensional case.

To estimate the smoothness of q_1 we will first prove the following result.

Lemma 2.1. *Assume that the potential $q(x)$ belongs to $L_{2\delta}^p(\mathbb{R}^2)$ with p and δ as in Proposition 1. Then the first nonlinear term (1.13) admits the integral representation*

$$q_1(x) = \frac{1}{8\pi^2} \left| \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} q(y) dy \right|^2. \quad (2.1)$$

Proof. Since (see [16])

$$\int_{S^1} e^{-ik(\vartheta, x-z)} d\vartheta = 2\pi J_0(|k||x-z|)$$

and

$$\vartheta e^{-ik(\vartheta, x-z)} = -\frac{1}{ik} \nabla_x e^{-ik(\vartheta, x-z)}$$

then using the following property for Bessel functions $(J_0(r))' = -J_1(r)$ we can rewrite the integrals in (1.13) with respect to k, ϑ and ϑ' in the equivalent form

as

$$4\pi^2 \int_{-\infty}^{\infty} |k| \tilde{G}_k^+(|y-z|) \left(J_0(|k||x-y|) J_0(|k||x-z|) - J_1(|k||x-y|) J_1(|k||x-z|) \frac{(x-z, x-y)}{|x-z||x-y|} \right) dk. \quad (2.2)$$

Due to the definition of \tilde{G}_k^+ (see (1.4) and (1.13)) this integral (2.2) can be rewritten as

$$-2\pi^2 \int_0^{\infty} k Y_0(k|z-y|) \left(J_0(k|x-y|) J_0(k|x-z|) - J_1(k|x-y|) J_1(k|x-z|) \frac{(x-y, x-z)}{|x-y||x-z|} \right) dk, \quad (2.3)$$

where Y_0 is the function of Neumann of 0 order.

In order to calculate (2.3) precisely it remains to calculate the following two integrals

$$\int_0^{\infty} k J_0(ak) J_0(bk) Y_0(ck) dk, \quad \int_0^{\infty} k J_1(ak) J_1(bk) Y_0(ck) dk, \quad (2.4)$$

where we denoted by $a = |x-y|$, $b = |x-z|$ and $c = |z-y|$. Let us remark that $a+b > c$ and $a-b < c$.

For the calculation of these two integrals we will use the following representation

$$Y_0(ck) = -\frac{2}{\pi} K_0(-ick) + iJ_0(ck),$$

where K_0 is the function of Macdonald of 0 order. With respect to such representation, in order to calculate the two integrals from (2.4) it is needed to calculate the following four integrals:

$$I_1 = \int_0^{\infty} k J_0(ak) J_0(bk) J_0(ck) dk, \quad I_2 = \int_0^{\infty} k J_0(ak) J_0(bk) K_0(-ick) dk$$

$$I_3 = \int_0^{\infty} k J_1(ak) J_1(bk) J_0(ck) dk, \quad I_4 = \int_0^{\infty} k J_1(ak) J_1(bk) K_0(-ick) dk.$$

Since a , b and c are the sides of a triangle we can apply formula (4) (see [16, p. 412]) and obtain

$$I_1 = \frac{1}{\pi ab} (1-X^2)^{-1/2}, \quad I_3 = \frac{X}{\pi ab} (1-X^2)^{-1/2}, \quad (2.5)$$

where $X = (a^2 + b^2 - c^2)/(2ab)$. Applying now formula (4) and (6) (see [16, p. 156, and p. 412, respectively]) and formulas (12) and (13) (see [1, p. 150]) we can obtain that

$$I_2 = \frac{i}{2ab} (1-X^2)^{-1/2}, \quad I_4 = \frac{iX}{2ab} (1-X^2)^{-1/2} - \frac{1}{2ab}, \quad (2.6)$$

where X is as in (2.5).

Combining (2.4)–(2.6) we obtain that the first integral in (2.4) is equal to zero (this fact has an independent interest) and the second integral is equal to

$$\frac{1}{\pi ab} = \frac{1}{\pi|x-y||x-z|}.$$

Therefore, this fact implies (see (1.13), (2.2)–(2.4)) that

$$\begin{aligned} q_1(x) &= \frac{1}{16\pi^3} \int_{\mathbb{R}^2 \times \mathbb{R}^2} q(y)q(z) \left(2\pi \frac{(x-y, x-z)}{|x-y|^2|x-z|^2} \right) dz dy \\ &= \frac{1}{8\pi^2} \left| \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} q(y) dy \right|^2. \end{aligned}$$

Thus, this lemma is proved. \square

Lemma 2.2. *Under the same assumptions for $q(x)$ as in Lemma 2.1 we have*

- 1) for $1 < p \leq 3/2$ the function $q_1(x)$ belongs to $(W_{p,-1}^1(\mathbb{R}^2))^2$;
- 2) for $3/2 < p < \infty$ the function $q_1(x)$ belongs to $(W_{p,2\delta-1}^1(\mathbb{R}^2))^2$ for $1 - 3/2p < 2\delta < 2 - 2/p$, where $W_{p,\sigma}^1(\mathbb{R}^2)$ denotes the space of all functions $f(x)$ from $L_p^p(\mathbb{R}^2)$ such that $\nabla f(x)$ belongs to $L^p(\mathbb{R}^2)$;
- 3) for $p = \infty$ the function $q_1(x)$ belongs to $(\Lambda^1(\mathbb{R}^2))^2$, where $\Lambda^1(\mathbb{R}^2)$ denotes the classical Zygmund space with smoothness index 1.

Proof. We introduce Riesz potential I^{-1} and Riesz transform R (see [15]) as

$$I^{-1}f(x) = F^{-1} \left(\frac{\hat{f}(\xi)}{|\xi|} \right) (x), \quad Rf(x) = F^{-1} \left(\frac{\xi \hat{f}(\xi)}{|\xi|} \right) (x),$$

where F^{-1} denotes the usual inverse Fourier transform in \mathbb{R}^2 . We first observe that $\nabla I^{-1} = R$ is bounded in $L^p(\mathbb{R}^2)$ for every $1 < p < \infty$ (see [15]). But due to the well-known estimates by Nirenberg and Walker (see [3]) we have also that

$$I^{-1} : L_{\sigma+1}^p(\mathbb{R}^2) \rightarrow L_{\sigma}^p(\mathbb{R}^2),$$

where $-2/p < \sigma < 2/p' - 1$. Since for $1 < p \leq 3/2$ we can choose $1 + \sigma = 0$ then the statements 1) and 2) of this lemma follow from these two facts immediately. As it concerns the case $p = \infty$ the statement 3) can be proved by exactly the same manner as Lemma 2.2 from [5]. Hence, this lemma is completely proved. \square

We are now in the position to formulate and prove our main result.

Theorem 1. *Assume that the potential $q(x)$ satisfies all conditions of Lemma 1.2. Then*

- 1) for $2 < p \leq \infty$ the difference $q_B(x) - q(x)$ is continuous;

- 2) for $3/2 < p \leq 2$ the difference $q_B(x) - q(x) - q_1(x)$ is continuous and bounded;
- 3) for $5/4 < p \leq 3/2$ the difference $q_B(x) - q(x) - q_1(x)$ belongs to the Sobolev space $H^t(\mathbb{R}^2)$ for any $t < 4 - 5/p$.

The proof of this theorem follows immediately from Lemmas 1.1, 1.2, 2.1 and 2.2 and we leave it for readers to check that the proof goes through.

Remark. This theorem gives that all singularities and jumps of the unknown potential from $L^p(\mathbb{R}^2)$ for $2 < p \leq \infty$ can be obtained exactly by the Born approximation. In particular, if the potential is the characteristic function of a bounded domain then this domain is uniquely determined by this scattering data. Whereas if the potential belongs to $L^p(\mathbb{R}^2)$ for $5/4 < p \leq 2$ then by the Born approximation we can obtain only the leading order singularities of the unknown potential.

Acknowledgement

We would like to thank A. Ruiz and J. A. Barselo for their observation that some steps in the proof of Lemmas 2.3 and 2.4 (in particular, the inequalities (2.19) and (4.13), respectively) in [7] are not correct and for useful discussions.

REFERENCES

1. H. Bateman, *Higher Transcendental Functions. Vol. 1*. McGraw-Hill Book Company, 1953.
2. M. S. Birman, On the number of eigenvalues in the quantum scattering problem. *Mat. Sbornik* (1960) **52**, 163–166.
3. L. Nirenberg and H. Walker, Null spaces of elliptic partial differential operators in \mathbb{R}^n . *J. Math. Anal. Appl.* (1973) **42**, 271–301.
4. P. Ola, L. Päivärinta, and V. Serov, Recovering singularities from backscattering in two dimensions. *Comm. PDE* (2001) **26**, 697–715.
5. L. Päivärinta and E. Somersalo, Inversion of discontinuities for the Schrödinger equation in three dimensions. *SIAM J. Math. Anal.* (1991) **22**, 480–499.
6. L. Päivärinta, V. Serov, and E. Somersalo, Reconstruction of singularities of a scattering potential in two dimensions. *Adv. Appl. Math.* (1994) **15**, 97–113.
7. L. Päivärinta and V. Serov, Recovery of singularities of a multidimensional scattering potential. *SIAM J. Math. Anal.* (1998) **29**, 697–711.
8. L. Päivärinta and V. Serov, New mapping properties for resolvent of the Laplacian and recovery of singularities of a multidimensional scattering potential. *Inverse Problems* (2001) **17**, 1321–1326.

9. L. Päivärinta and V. Serov, An n -dimensional Borg–Levinson theorem for singular potentials. *Adv. Appl. Math.* (2002) **23**, 509–520.
10. L. Päivärinta and V. Serov, New estimates of the Green–Faddeev function and recovering of singularities in the two-dimensional Schrödinger operator with fixed energy. *Inverse Problems* (2005) **21**, 1291–1301.
11. A. G. Razborov and V. S. Serov, The spectrum of the Schrödinger operator with Kato potential. *Diff. Uravneniya* (2000) **36**, 689–693 (in Russian); *Diff. Eq.* (2000) **36**, 767–772 (in English).
12. V. S. Serov, A scattering problem for the Schrödinger operator with singular potential in two-dimensional case I. *Diff. Uravneniya* (1990) **26**, 851–860.
13. V. S. Serov, A scattering problem for the Schrödinger operator with singular potential in two-dimensional case II. *Diff. Uravneniya* (1991) **27**, 120–128.
14. V. S. Serov, Recovering the singularities of a potential in two-dimensional Schrödinger operator with fixed angle scattering data. *Rus. Math. Dokl.* (2002) **358**, 160–162.
15. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton, 1970.
16. G. N. Watson, *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, Cambridge, 1948.