Inverse Problems for Time Harmonic Electrodynamics

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Introduction

In his famous article *A Dynamical Theory of Electromagnetic Field* of 1864 James Maxwell wrote down differential equations that describe the laws of electromagnetism in full generality. The four equations of Maxwell,

$$\nabla \cdot \mathbf{D}(x,t) = \rho(x,t), \tag{1}$$

$$\nabla \cdot \mathbf{B}(x,t) = 0. \tag{2}$$

$$\frac{\partial \mathbf{B}(x,t)}{\partial t} + \nabla \times \mathbf{E}(x,t) = 0, \tag{3}$$

$$-\frac{\partial \mathbf{D}(x,t)}{\partial t} + \nabla \times \mathbf{H}(x,t) = \mathbf{J}(x,t), \tag{4}$$

describe the dynamics of the five vector fields **E**,**D**,**B**, **H** and **J**. Here $\mathbf{E}(x,t)$ is the electric field, $\mathbf{D}(x,t)$ the *electric induction*or *electric flux density*, **B** the *magnetic induction* or *magnetic flux density*, $\mathbf{H}(x,t)$ is the *magnetic field* and finally $\mathbf{J}(x,t)$ is the electric current density. Since modern vector calculus was unknown to Maxwell, he formulated these equations as 20 scalar equation. The above present form of these equations originates from Oliver Heaviside from the 1880's.

Let us now briefly explain the nature of each equation: Equation (1) is the Gauss law and it tells that infinitesimally the total flow of the electric induction is equal to the density of free charges. The scalar field ρ here is the *charge density.* Equation (2) is the magnetic analogue of the Gauss law saying that there are no free magnetic charges. Equation (3) called Faraday's law explains how a changing magnetic flux creates an electric current in a conductive loop;

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a law that is based on a series of experiments that Faraday was performing in the years 1831 and 1832. That time the dual phenomenon to Faraday's law of induction was known as Ampère's law. It explains how an electric current in a loop creates a magnetic field and in our notation reads as

$$\nabla \times \mathbf{H}(x,t) = \mathbf{J}(x,t). \tag{5}$$

The asymmetry in the equations (3) and (5) worried Maxwell and he started to think about Faraday's idea of polarization. Under influence of an electric field a medium starts to polarize. This means a little change in the position of charges and hence an electric current. He added a new current term $\frac{\partial \mathbf{D}(x,t)}{\partial t}$ to Ampère's law and as a result of purely theoretical reasoning discovered among other things electromagnetic waves. The existence of these waves was later verified by the experiments of Herz.

We call equations (1)-(4) macroscopic, because they deal directly with observable physical quantities and explain how they are related to each other. In particular the structure of media is of no consequence. Also without any additional assumptions they are not enough to determine the fields uniquely as a moments thought will reveal. Besides of his four differential equations Maxwell described four so called *structure equations* or *constitutive equations* that relate **E** and **D**, **B** and **H** and finally **J** and **E**:

$$\mathbf{D}(x,t) = \epsilon(x)\mathbf{E}(x,t), \tag{6}$$

$$\mathbf{B}(x,t) = \mu(x)\mathbf{H}(x,t) \tag{7}$$

$$\mathbf{J}(x,t) = \sigma(x)\mathbf{E}(x,t). \tag{8}$$

Here $\epsilon(x)$ is the *electric permitivity* or *dielectricity*, $\mu(x)$ the *magnetic permeability* and $\sigma(x)$ the *electric conductivity*. Roughly speaking, under influence of an external electric field ϵ tells us the magnitude in which the material is tempting to form electric dipoles and conductivity the magnitude of the electric current . The permeability μ is analog to ϵ. It explains the magnitude in which the material is forming magnetic dipoles ie. small circular loops in an external magnetic field.

The goal of an electromagnetic inverse problem is to determine these parameters inside an unknown object in a non-invasive way, say from boundary measurements. The applications arise naturally in geophysical prospecting, nondestructive testing and medical imaging. As an example we mention here the problem of detecting leukemia by using electromagnetic waves. This is made possible by the fact that leukemia causes a change of electric permitivity in the bone marrow by factor two. For more details we refer to [3] and Chapter 2 in [1].

In this article we review the uniqueness results and reconstruction algorithms for time-harmonic fixed frequency inverse problems. This means that the time dependence of all fields is assumed to be $e^{-i\omega t}$ and we fix the frequency ω.

Instead of describing the electromagnetic fields as vector fields in an euclidean space we have chosen define them as differential forms in a riemannian manifold.This does not only give additional generality but also clarifies the nature of different physical fields. As an example the electric and magnetic induction above have a physically well defined flux through a surface, hence they are integrable over two dimensional surfaces and consequently they are two forms. The formulation using forms makes obvious the invariance properties of Maxwell's equations. At the same time some formulas, like the radiation condition, are considerably simplified.

The structure of the article is as follows. In the first two sections we describe the problems to be considered starting from Maxwell's equations and set up the mathematical framework that is going to be used. We also offer references to aspects of the problem that are not covered in detail in these notes and give some historical background.

In the third section we rescale Maxwell's equations and then complete them into a Dirac type elliptic system. For a similar time domain formulation for Maxwell's equations see [9]

In section four we describe how the boundary data describing our idealized measurement, i.e., the equivalent of the *Dirichlet–to–Neumann* map of the *electrical impedance tomography* problem, determines the boundary values and the normal derivatives at the boundary of the unknown material parameters. To be able to see into the interior, one needs a suitable family of solutions (and a comaptible integration by parts), and these are described in section five.

To keep the reconstruction algrorithm constructive (at least mathematically) the next step is to show that our boundary data makes it possible to determine the Cauchy–data of these special solutions, and this is done in section six. In the next two sections we explain how the unique determination of the parameters is proved.

1 Time-harmonic Maxwell's equations

Let us assume that the time dependence of all fields in (1)-(4) is harmonic with frequency $\omega > 0$, i.e. all time dependent fields above are of the form $f(x,t) = e^{-i\omega t} f(x)$. Then cancelling out the oscillatory exponential we end up with the following system:

$$i\omega\,\mathbf{D}(x) + \nabla \times \mathbf{H}(x) = \mathbf{J}(x), \tag{9}$$

$$\nabla \cdot \mathbf{D}(x) = \rho(x), \tag{10}$$

$$-i\omega\,\mathbf{B}(x) + \nabla \times \mathbf{E}(x) = 0, \tag{11}$$

$$\nabla \cdot \mathbf{B}(x) = 0. \tag{12}$$

Let us interprete the electric field and magnetic field as one–forms by identifying a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ with the one form (we are using the Einstein summation convention whenever convenient) $F = F_i dx^i$. To interprete the above equations in terms of forms we also have to identify vector fields with two forms and this can be done as follows: the vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is identified with the two form $F_1\,(dx^2 \wedge dx^3) + F_2\,(dx^3 \wedge dx^1) + F_3(dx^1 \wedge dx^2)$, or invariantly this can be expressed in terms of the euclidean Hodge–star operator $*_e$ as $*_e F$. The equations (9)–(12) now take the form

$$i\omega D(x) + d\,H(x) = J(x), \tag{13}$$

$$d\,D(x) = \tilde{\rho}(x), \tag{14}$$

$$-i\omega\,B(x) + d\,E(x) = 0, \tag{15}$$

$$d\,D(x) = 0. \tag{16}$$

where $D = *_e D$, $B = *_e B$, $\tilde{\rho} = \rho dV_e$ and $dV_e = dx^1 \wedge dx^2 \wedge dx^3$ is the euclidean volume element.

For the moment we consider the above system on an arbitrary smooth differentiable and orientable three–manifold M, with or without a boundary.

The microscopic structure of medium in the domain is modelled by introducing a Riemannian metric g on M, and postulating that the magnetic and electric inductances (which are two–forms) are related to the magnetic and electric fields via

$$D = \gamma(x) * E, \quad B = \mu(x) * H. \tag{17}$$

Here $*$ is the Hodge–star operator with recpect to the metric g and μ and γ are smooth scalar functions,

$$\gamma(x) = \varepsilon(x) + i\frac{\sigma(x)}{\omega}$$

with ε and μ are equal to constants ε_0 and μ_0 respectively outside a compact set, both are bounded and strictly positive, and σ is a non–negative compactly supported function. In this formulation, the Ohmic part of the current density J is merged in the electric induction D, and J in (13) represents the forced current densities. Note that the Maxwell.s equations (13)–(16) are purely topological ie. there is no reference to the underlying metric. The metric properties appear as expected in the constitutive equations (17). As Keppler said in his thesis on 1602 *'Ubi materia , ibi geometria'* ie where is matter, there is geometry.

In the sequel, we shall assume throughout that in the domain M of interest,

$$\tilde{\rho} = 0, \quad J = 0.$$

We arrive at Maxwell's equations for the so called *perfect media*,

$$d\,H(x) + i\omega\gamma(x) * E(x) = 0, \tag{18}$$

$$d\,\gamma * E(x) = 0, \tag{19}$$

$$d\,E(x) - i\omega\mu(x) * H(x) = 0, \tag{20}$$

$$d\,\mu * H(x) = 0. \tag{21}$$

We remark that in reality not all media obey the constitutive relations (**??**) used here. First of all, in some applications the functions $\mu(x)$ and $\gamma(x)$ also depend on the freguency variable via so called *dispersion relations*. In the time domain, the frequency dependency corresponds to *memory* of the matter, i.e., the response of the material such as the polarization are not instantaneous but depend on the past values of the fields. Mathematically, this means that in the time domain, the constitutive relations become causal time convolutions. Secondly, not all media is *isotropic*: the media is isotropic if one can choose $\gamma(x)$ and $\mu(x)$ to be scalar functions. For example muscle tissue is anisotropic and these functions have to be allowed to be more general tensors. Finally, the constitutive relations might be more complicated: for example both D and B might depend on a linear combination of E and H, which leads to *chiral media*, or the dependence could be nonlinear. Materials with this behaviour are in abundance in nature, for example several crystals are chiral, and this fact is used to create artifical sweeteners, and metals in strong magnetic fields behave in a nonlinear fashion.

In this work we limit ourselves mostly to perfect media and will refer the reader interested in the more general cases to the literature cited in references.

2 Inverse problems

In this section we formulate the inverse boundary value problem as well as the inverse scattering problems for Maxwell's equations.

We start by fixing certain notations. Assume first that M is a smooth compact oriented 3–manifold with $\partial M \neq \emptyset$. We denote by $\Omega^k M$, $0 \leq k \leq 3$ the vector bundle of smooth k–forms on M. Let $i : \partial M \to M$ denote the canonical imbedding. We define the *tangential trace* of k–forms as

$$\mathbf{t} : \Omega^k M \to \Omega^k \partial M,\ \mathbf{t}\omega = i^*\omega, \quad \omega \in \Omega^k M,\ 0 \leq k \leq 2.$$

where i^* is the pull-back of i. Similarly, we define the *normal trace* as

$$\mathbf{n} : \Omega^k M \to \Omega^{3-k} \partial M,\ \mathbf{n}\omega = i^*(*\omega), \quad \omega \in \Omega^k M,\ 1 \leq k \leq 3.$$

Observe that for 1–forms, the tangential component corresponds to the tangential component of the vector field while for 2–forms, it corresponds to the transversal flux through the boundary. For the normal trace, the roles are interchanged. For more precise discussion, see e.g. [20]. (In fact, the definition of the normal trace here differs from that given in the cited article.)

The Stokes formula can be written now as follows: Let $\delta : \Omega^k M \to \Omega^{k-1}M$ denote the codifferential for k–forms,

$$\delta = (-1)^{n(k+1)+1} * d* = (-1)^k * d * \text{ for dimension } n = 3.$$

By denoting the inner product of k–forms over M as

$$(\omega, \eta) = \int_M \omega \wedge *\overline{\eta},$$

while at the boundary we denote

$$\langle \omega, \eta \rangle = \int_{\partial M} \omega \wedge \overline{\eta}, \quad \omega \in \Omega^k \partial M,\ \eta \in \Omega^{2-k}\partial M.$$

We have the identity

$$(d\omega, \eta) - (\omega, \delta\eta) = \langle \mathbf{t}\omega, \mathbf{n}\eta \rangle, \quad \omega \in \Omega^k M,\ \eta \in \Omega^{k+1}M \tag{22}$$

With these notations, let us define the *admittance map* for Maxwell's equations at the boundary: Assume for simplicity that $\gamma - \epsilon_0, \mu - \mu_0 \in C_0^\infty(\mathrm{int}(M))$, i.e., the material parameters near the boundary ∂M are constants $\epsilon_0 > 0$ and $\mu_0 > 0$, respectively. We define

$$\Lambda : \mathbf{t}(\epsilon_0^{1/2} E) \mapsto \mathbf{t}(\mu_0^{1/2} H).$$

The inverse boundary value problem (IBP) we consider here can be stated as follows:

IBP: From the knowledge of the admittance map Λ at the boundary, determine the material parameters γ and μ in M.

One can also consider this problem for chiral media, and for this the interested reader is referred to [11]. For general anisotropic media it is not even clear what the right conjecture is, but if γ and μ are conformally related to each other, then the linarization suggests that the non–uniqueness arises from diffeomorphisms of M to itself fixing the boundary, see [22].

Equally natural is the inverse scattering problem. For simplicity, we assume here that $M = \mathbb{R}^3$ endowed with a metric that coincides with the euclidean one outside some smooth bounded set D, and furthermore, $\epsilon(x) = \epsilon_0$ and $\mu(x) = \mu_0$ for $x \notin D$. Consider the following plane–wave solution of the euclidean Maxwell's equations. Let

$$E_{\mathrm{i}}(x) = e^{i\langle x, k\rangle} p, \quad H_{\mathrm{i}}(x) = e^{i\langle x, k\rangle} q$$

be solutions of Maxwell's equations in vacuum, where k satisfies $|k|^2 = \epsilon_0 \mu_0 \omega^2$. To satisfy equations (18) and (20), we require that the polarization 1–forms p and q satisfy

$$k \wedge p = \omega \mu_0 *_e q, \quad k \wedge q = -\omega \epsilon_0 *_e p,$$

where we have identified the vector k with a 1-form through $k(v) = \langle k, v\rangle$. It follows then that

$$\mu_0\|q\|^2 = \mu_0 q \wedge *_e q = \epsilon_0\|p\|^2, \quad p \wedge *_e q = 0,$$

and furthermore, the equations (19) and (21) require that

$$k \wedge *_e p = 0, \quad k \wedge *_e q = 0.$$

The total field is written as a sum of the incoming field above plus the scattered field,

$$E = E_{\rm i} + E_{\rm sc}, \quad H = H_{\rm i} + H_{\rm sc}.$$

The scattered field needs to satisfy a radiation condition at infinity. To understand better the the radiation condition for differential forms, let us go back for a while to the physical time domain picture. Furthermore, let us consider for simplicity for a while just electromagnetic field in vacuum. In the discussion below, we write concisely $E = E(x,t)$ and $H = H(x,t)$ for the physical time domain fields in vacuum. Consider a ball B_R of radius $R > 0$. The total energy of the field within the ball is expressed as

$$\mathcal{E} = \epsilon_0\|E\|_R^2 + \mu_0\|H\|_R^2 = \mathcal{E}_E + \mathcal{E}_H,$$

where we wrote

$$\|E\|_R^2 = \int_{B_R} E \wedge *_e E = (E, E)_R.$$

Consider the electric part of the energy. The time derivative of it gives us

$$\frac{\mathcal{E}_E}{\partial t} = 2\epsilon_0 \left(\frac{\partial E}{\partial t}, E\right)_R = \frac{2}{\mu_0}(\delta B, E)_R,$$

and further, by applying Stokes law,

$$\frac{\mathcal{E}_E}{\partial t} = \frac{2}{\mu_0}(B, dE)_R + \frac{2}{\mu_0}\langle \mathbf{n}B, \mathbf{t}E\rangle_R. \tag{23}$$

Again, from Maxwell's equations, we obtain

$$\frac{2}{\mu_0}(B, dE)_R = -\frac{2}{\mu_0}\left(B, \frac{\partial B}{\partial t}\right) = -\frac{\partial \mathcal{E}_H}{\partial t}.$$

By substituting this identity into equation (23), we find that the total change of energy in the ball equals the flux through the boundary ∂B_R, i.e.,

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{\partial \mathcal{E}_E}{\partial t} + \frac{\partial \mathcal{E}_H}{\partial t} = \frac{2}{\mu_0}\langle \mathbf{n}B, \mathbf{t}E\rangle_R = \langle \mathbf{t}H, \mathbf{t}E\rangle_R.$$

The radiation conditions are now defined in such a way that for large R, the energy flux either becomes negative (outgoing waves) or positive (incoming wave). For the outgoing wave, we write

$$\langle \mathbf{t}H, \mathbf{t}E\rangle_R = -\left(\frac{\epsilon_0}{\mu_0}\right)^{1/2}\langle \mathbf{t}E, \mathbf{t}E\rangle_R = \left\langle \mathbf{t}H + \left(\frac{\epsilon_0}{\mu_0}\right)^{1/2}\mathbf{t}E, \mathbf{t}E\right\rangle_R.$$

Hence, to assure that the last term has asymptotically vanishing effect, we set the radiation condition

$$\mathbf{t}(\epsilon_0^{1/2}E + \mu_0^{1/2}H) = o\left(\frac{1}{|x|}\right). \tag{24}$$

This is the outgoing *radiation condition* for differential forms that we impose for the scattered field in the frequency domain. Compared with the *Silver-Müller radiation condition* in the vector formalism this appears strikingly simple.

By using the representations of the scattered fields in terms of Green's functions, it is possible to derive an asymptotic representation of the fields,

$$\mathbf{t}E_{\rm sc}(x) = E_\infty(\hat{x}; k; p)\frac{e^{i|k||x|}}{|x|} + o(|x|^{-1}), \tag{25}$$

$$\mathbf{t}H_{\rm sc}(x) = H_\infty(\hat{x}; k; p)\frac{e^{i|k||x|}}{|x|} + o(|x|^{-1}), \tag{26}$$

where $\hat{x} = x/|x|$ and the mutually orthogonal one–forms (defined on the unit sphere) E_∞ and H_∞ are called respectively the electric– and magnetic far–field patterns corresponding to the polarization p and incidence direction k with $|k|^2 = \epsilon_0\mu_0\omega^2$. Note that one only needs to specify one of these, the other one can then be immediately obtained from the radiation conditions. The inverse scattering problem (ISP) can now be formulated as follows.

ISP: From the knowledge of $E_\infty(\theta;\omega,p)$ for all $\theta,\omega \in S^2$ and for three linearly independent polarizations p determine the material parameters γ and μ.

If one knows the admittance map on a smooth surface Γ enclosing the inhomogenuity, then the boundary value on Γ of the rescaled scattered field $e_{\rm sc} = \gamma^{1/2}E_{\rm sc}$ corresponding to the incoming plane wave with electric component $e_{\rm i} = \varepsilon_0^{1/2}pe^{i\langle x,k\rangle}$ can be solved from the boundary integral equation (argument is like the one used to derive (49) in section 6)

$$\frac{1}{2}\mathbf{t}e_{\rm sc} = \mathbf{t}e_{\rm i} + D_k\Lambda\mathbf{t}e_{\rm sc} - K_k\mathbf{t}e_{\rm sc}, \tag{27}$$

on Γ, here operators D_k and K_k are defined analogously to D and K introduced in Section 6 but using the standard outgoing fundamental solution $-e^{ik|x|}/4\pi|x|$ instead of the Faddeev Green's function. The mapping properties are unchanged by this, and the unique solvability follows in a standard manner assuming that ω is not a resonance frequency. Hence we know the tangential boundary values of e and h on Γ, and thus also the far–fields are determined by the impedance map.

To be able to reduce the ISP to IBP, we however need the opposite direction, and this is not as simple. The difficulty (and also its resolution) are similar to the acoustic case, so we'll be rather brief in describing it. If one knows the far–fields of all waves scattered by plane waves, one also knows using the Rellich–argument the scattered fields in the complement of the union of the supports of $\gamma - \varepsilon_0$ and $\mu - \mu_0$, and thus one also knows their boundary values on Γ, i.e. one knows the restriction of the impedance map to all total fields corresponding to incoming plane–waves. The problem is to show that this data determines the impedance map. This was shown to be true for Maxwell's equations in [19], Sect. 6.4., and hence the scattering problem is reduced to IBP. The argument used is a modification of the idea of Nachman ([13]) and Ramm ([18]), and actually works for the full Picard–system, hence effectively dealing with the acoustic and electromagnetic cases simultaneously. Using this argument one however has to assume that the interface Γ is chosen so that ω is not a magnetic resonance frequency. Of course one is free to choose the interface so that this is avoided, but in practice it is not easy to determine when one is close to an resonance frequency.

One can also deal with the inverse scattering problem directly, without reducing it to the IBP: for Maxwell's equations this was done D. Colton and L. Päivärinta in [4] assuming that $\mu = \mu_0$, and stability results were obtained in [5]. The crucial part is again the construction of the exponentially increasing solutions, and in these proofs one doesn't need any assumptions on ω.

3 Scaled system

In this section we follow the idea of [17] and rescale the electromagnetic forms in such a way that we only have to deal with one metric and complete this system to an elliptic system of Dirac–type. In [17] this was only done for the case of euclidean background metric, but the principle remains the same in this more general setting. The basic idea is of course old and well–known: Eventhough the divergence conditions are implied by the two other equations they make it possible to reduce the system to an elliptic system that coincides with the Helmholtz equation for **E** and **H** respectively in the constant parameter case. Also, the divergence conditions are crucial when analyzing the low–frequency limit, since they single out the right limit value (remember that Maxwell's equations have an infinite dimensional kernel when $\omega = 0$). The approach we follow is a modification of this, originally due to R. Picard (see [15]). The idea is to get a first order elliptic and symmetric system (at least in the principal part) that under some conditions reduces to Maxwell's equations. The ellipticity is achieved by including the divergence conditions to the system, but to make it symmetric one needs to modify it further. So, lets start with system (18)–(21) and introduce 3–forms Φ and Ψ by

$$i\omega\Phi = d(\gamma * E), \quad i\omega\Psi = d(\mu * H). \tag{28}$$

Of course, if E and H satisfy Maxwell's equations, these forms vanish. Now we modify (18) and (20) to

$$dH - \frac{1}{\mu} * d * (\frac{1}{\gamma}\Phi) + i\omega\gamma * E = 0, \tag{29}$$

$$dE + \frac{1}{\gamma} * d * (\frac{1}{\mu}\Psi) - i\omega\mu * H = 0. \tag{30}$$

The principal part of this system still depends on γ and μ, and in order to make it more simple we scale the unknown fields: Let

$$e = \gamma^{1/2}E, \quad h = \mu^{1/2}H, \quad \phi = \frac{1}{\gamma\mu^{1/2}}\Phi, \quad \psi = \frac{1}{\gamma^{1/2}\mu}\Psi. \tag{31}$$

This makes the physical dimensions of the unknown 1– and 3–forms equal, and it also makes the principal part of the system to depend only on the background metric g as we shall see. To get a full graded algebra, let us further define the 0– and 2–forms φ and b as

$$\varphi = *\phi, \quad b = *h.$$

We introduce the notation

$$\mathbf{\Omega}M = \Omega^0 M \times \Omega^1 M \times \Omega^2 M \times \Omega^3 M,$$

for the full exterior bundle and endow $\mathbf{\Omega}M$ with the obvious inner product: If $u = (u^0, u^1, u^2, u^3)$, $v = (v^0, v^1, v^2, v^3) \in \mathbf{\Omega}M$, we set

$$(u, v) = \sum_{j=0}^{3} \int_M u^j \wedge *\overline{v^j}.$$

Let us define a graded form

$$X = (\varphi, e, b, \psi) \in \mathbf{\Omega}M.$$

A straightforward insertion into the augmented equations (28)–(30) along with the identities $\delta = (-1)^k * d*$ and $** = (-1)^{k(n-k)} = 1$ for forms of degree k in $\mathbb{R}^n$ with $n = 3$ give that X satisfies the system

$$(P + i\kappa(x))X + VX = 0, \tag{32}$$

where the principal part is

$$P = \begin{pmatrix} 0 & -\delta & 0 & 0 \\ d & 0 & -\delta & 0 \\ 0 & d & 0 & -\delta \\ 0 & 0 & d & 0 \end{pmatrix},$$

the scalar $\kappa(x)$ is the non–constant wave number,

$$\kappa(x) = \omega(\gamma(x)\mu(x))^{1/2},$$

and V is a local potential given by

$$V = \left(\begin{array}{cccc} 0 & i\langle \alpha, \cdot \rangle & 0 & 0 \\ i(*\cdot) * \beta & 0 & i\beta \wedge \cdot & 0 \\ 0 & -i\alpha \wedge \cdot & 0 & i(*\cdot) * \alpha \\ 0 & 0 & i\langle \beta, \cdot \rangle & 0 \end{array} \right).$$

where

$$\alpha = \frac{1}{2} \ln \gamma, \quad \beta = \frac{1}{2} \ln \mu.$$

The first order operator has several important properties. First, Stokes formula (22) implies that

$$(Pu, v) + (u, Pv) = \langle \mathbf{t}u, \mathbf{n}v \rangle + \overline{\langle \mathbf{t}v, \mathbf{n}u \rangle},$$

where we introduced the shorthand notation $\mathbf{t}v$, $\mathbf{n}v \in \mathbf{\Omega}\partial M = \Omega^0 \partial M \times \Omega^1 \partial M \times \Omega^2 \partial M$,

$$\mathbf{t}v = (\mathbf{t}v^0, \mathbf{t}v^1, \mathbf{t}v^2), \quad \mathbf{n}v = (\mathbf{n}v^3, \mathbf{n}v^2, \mathbf{n}v^1),$$

and

$$\langle \mathbf{t}u, \mathbf{n}v \rangle = \langle \mathbf{t}u^0, \mathbf{n}v^1 \rangle + \langle \mathbf{t}u^1, \mathbf{n}v^2 \rangle + \langle \mathbf{t}u^2, \mathbf{n}v^3 \rangle.$$

Observe that this expression does not define an inner product on ∂M.

Second, we observe immediately that

$$P^2 = -\mathbf{\Delta} = -\mathrm{diag}(\Delta^0, \Delta^1, \Delta^2, \Delta^3),$$

where $\Delta^k = \delta d + d\delta$ is the Laplace-Beltrami operator for k–forms. But more is true. Namely, splitting the Grassmann–algebra into its even– and odd–degree parts

$$\mathbf{\Omega} M = \mathbf{\Omega}^+ M \oplus \mathbf{\Omega}^- M$$

we can write the potential as an off-diagonal block matrix

$$V = \left(\begin{array}{cc} 0 & V_- \\ V_+ & 0 \end{array} \right) = i \left(\begin{array}{cccc} 0 & 0 & \langle \alpha, \cdot \rangle & 0 \\ 0 & 0 & -\alpha \wedge \cdot & (*\psi) * \alpha \\ \beta & -\iota_\beta \cdot & 0 & 0 \\ 0 & \langle \beta, \cdot \rangle * & 0 & 0 \end{array} \right),$$

and the Picard–operator becomes just the Dirac–type operator $\mathbf{D} = d - \delta$. Hence we can write (32) equivalently as

$$(\mathbf{D} + i\kappa + V)X = 0.$$

Let now

$$\tilde{V} = \left(\begin{array}{cc} 0 & \tilde{V}_- \\ \tilde{V}_+ & 0 \end{array} \right) = i \left(\begin{array}{cccc} 0 & 0 & \langle \beta, \cdot \rangle & 0 \\ 0 & 0 & -\beta \wedge \cdot & \iota_\beta \\ \alpha & -\iota_\alpha \cdot & 0 & 0 \\ 0 & \alpha \wedge \cdot & 0 & 0 \end{array} \right), \tag{33}$$

i.e. $\tilde{V}$ is the adjoint of V. Then we have the following proposition:

Lemma 3.1 *The first order terms of the product*

$$(\mathbf{D} + i\kappa(x) + V(x))(\mathbf{D} - i\kappa(x) - \tilde{V}(x)) \tag{34}$$

vanish, i.e. it is of the form $-\Delta + k^2 + Q(x)$*, where* Q *is a zeroth order pointwise multiplier.*

Proof. The non trivial part of the lemma is of course the vanishing of the first order derivatives of the commutator–like term $\mathbf{D}\tilde{V} - V\mathbf{D}$. Since both $\mathbf{D}$ and V change the parity of the degree, we may consider only the even degree part. The odd–degree case is handled similarly. By a direct computation we get for $u_+ = u_0 + u_2$ that

$$(d-\delta)\tilde{V}_+ u_+ = \begin{pmatrix} -\delta(\alpha u_0) + \delta(\iota_\alpha u_2) \\ d(\alpha u_0) - d(\iota_\alpha u_2) - \delta(\alpha \wedge u_2) \end{pmatrix}$$

and

$$V_-(d-\delta)u_+ = \begin{pmatrix} \langle \alpha, du_0 - \delta u_2 \rangle \\ -\alpha \wedge du_0 + \alpha \wedge \delta u_2 + (*\alpha) * du_2 \end{pmatrix}$$

from which the claim follows for a straightforward computation done using a local orthonormal co–frame shows the differences

$$\langle \alpha, du_0 - \delta u_2 \rangle - (-\delta(\alpha u_0) + \delta(\iota_\alpha u_2),$$

$$-\alpha \wedge du_0 - d(\alpha u_0)$$

and

$$\alpha \wedge \delta u_2 + (*\alpha) * du_2 - (-d(\iota_\alpha u_2) - \delta(\alpha \wedge u_2))$$

are of order zero in u_+. □

4 Green's function

In this section we derive the exponentially growing (or Faddeev's) Greens function for the complete Maxwell system treated in the previous section. However, we have to confine ourselves in the Euclidian case in the sequel. Thus, we shall assume that $g = g_e$ is the Euclidian metric, and the euclidian normal coordiantes are denoted by (x^1, x^2, x^3).

We start by recaling the definition of the scalar Faddeev's Green's function: For any $\zeta \in \mathbb{C}^3$, set

$$G(x) = G_\zeta(x) = e^{i\langle x, \zeta \rangle} g_\zeta(x), \quad g_\zeta(x) = \left(\frac{1}{2\pi}\right)^3 \int_{\mathbb{R}^3} \frac{e^{i\langle x, \xi \rangle}}{|\xi|^2 + 2\langle \xi, \zeta \rangle} d\xi,$$

where the inner products are the real inner products, i.e., no complex conjugation is included. When ζ is chosen in such a way that

$$\langle \zeta, \zeta \rangle = k^2, \tag{35}$$

the function G is indeed Green's function for the Helmholtz– operator,

$$(\Delta - k^2)G(x) = \delta(x).$$

Note that our Δ is now the geometer's Laplacian $d\delta + \delta d$, which is a positive operator.

This scalar Green's function has the following important asymptotic property as $|\zeta| \to \infty$ along the variety $\{\zeta \in \mathbb{C}^3; \langle\zeta, \zeta\rangle = k^2\}$. Letting L^2_δ be the weighted L^2–space with norm

$$||f||^2_\delta = \int_{\mathbb{R}^3} |f|^2(1 + |x|^2)^\delta \, dx \tag{36}$$

we have the following estimate due to Sylvester and Uhlmann ([23]):

Proposition 4.1 *For $|\zeta|$ large, we have*

$$\|g_\zeta * f\|_\delta \leq \frac{C}{|\zeta|}||f||_{\delta+1}$$

By using this scalar Green's function, we define an exponentially growing Green's tensor for $\mathbf{\Delta} - k^2$ by setting

$$\begin{aligned}
\mathbf{G}(x-y) &= G(x-y)(1, \sum_{j=1}^{3} dx^j \otimes dy^j, \sum_{j=1}^{3} \theta_j \otimes \nu_j, dV_x \otimes dV_y) \\
&= G(x-y)\mathbf{I},
\end{aligned}$$

where $\theta_j = (1/2)\varepsilon_{jk\ell}dx^k \wedge dx^\ell$ and $\nu_j = (1/2)\varepsilon_{jk\ell}dy^k \wedge dy^\ell$ and $dV_x = dx^1 \wedge dx^2 \wedge dx^3$, $dV_y = dy^1 \wedge dy^2 \wedge dy^3$. Oberve that

$$\mathbf{I} \wedge *(\lambda_j dy^j) = \lambda_j dx^j.$$

For later reference, let us note that $\mathbf{I}$ can be written componentwise as

$$\mathbf{I} = \sum_{j=1}^{8} \omega^j_x \otimes \omega^j_y, \tag{37}$$

where $\omega^1_x = \omega^1_y = (1, 0, 0, 0)$, $\omega^2_x = (0, dx^1, 0, 0)$, $\omega^2_y = (0, dy^1, 0, 0)$ and so on.

With the help of this Green's tensor, we define now a graded form that could be called a *generalized Sommerfeld potential.* Let $Y_0 \in \mathbf{\Omega}M$ be any graded form satisfying

$$(-\mathbf{\Delta} + k^2)Y_0 = 0 \tag{38}$$

in $\mathbb{R}^3$. We seek to solve the potential $Y \in \mathbf{\Omega}M$ from the Lippmann–Schwinger type equation

$$Y(x) = Y_0(x) - \int_M \mathbf{G}(x-y) \wedge *(Q(y)Y(y)). \tag{39}$$

The existence of such solution for large $|\zeta|$ is guaranteed by Proposition 4.1. Also, we observe that Y satisfies the Schrödinger equation

$$(-\mathbf{\Delta} + k^2 + Q(x))Y(x) = 0. \tag{40}$$

¿From Proposition 4.1, we obtain also the important information of the asymptotic behaviour of Y for large $|\zeta|$.

Theorem 4.2 *For $|\zeta|$ large enough, $-1 < \delta < 0$, and for any constant coefficient form y_0 which is bounded in ζ, the equation (39) has a unique solution $Y = e^{i\langle x,\zeta\rangle}(y_0 + w_\zeta)$, where $\|w_\zeta\|_\delta < C/|\zeta|$.*

For later use, we fix already here the form Y_0 and require that it is of the form

$$Y_0(x) = e^{i\langle x,\zeta\rangle} y_0,$$

where we assume that $\zeta \in \mathbb{C}^3$ satisfies the condition (35), guaranteeing that the equation () to be valid. Furthermore, the constant graded form $y_0 = (y^0, y^1, y^2, y^3)$ is required to satisfy

$$ky^0 = *(\zeta_j dx^j \wedge *y^1), \quad ky^3 = \zeta_j dx^j \wedge y^2,$$

so that

$$((P - ik)Y_0)^0 = 0, \quad ((P - ik)Y_0)^3 = 0, \tag{41}$$

i.e., the 0–form and 3–form components of $(P - ik)Y_0(x)$ vanish.

Now we use the decomposition property (3.1). By setting

$$X(x) = (P - i\kappa(x) + \tilde{V}(x))Y(x), \tag{42}$$

we find that X satisfies the complete Maxwell system

$$(P + i\kappa(x) + V(x))X(x) = 0.$$

We call this solution the *exponentially growing solution* of the complete Maxwell system.

¿From the above definition, it is not obvious that X is indeed a solution to the original Maxwell system, i.e., that $x^0 = 0$ and $x^3 = 0$ as they should in order that the pair (x^1, x^2) would represent scaled electric and magnetic fields. However, one can prove the following result.

Lemma 4.3 *Assume that Y_0 is chosen so that the conditions (41) are satisfied. Then, for large $|\zeta|$, we have $x^0 = 0$ and $x^3 = 0$.*

Proof. In view of what was said above, it only remains to check that for $|\zeta|$ large enough the scalar and three form components of X_ζ vanish. Let's show this for the first component, the last component being handled similarly. It is a simple but somewhat tedious matter to check that the zero–order part $Y_{\zeta,0}$ satisfies the *scalar* Schr?dinger equation

$$-(\Delta + k^2)Y_{\zeta,0} + qY_{\zeta,0} = 0 \tag{43}$$

where the potential $q(x)$ is given by

$$q = (\kappa^2 - k^2) + \frac{\Delta\mu^{1/2}}{\mu^{1/2}} - 2\left|\frac{\nabla\mu^{1/2}}{\mu^{1/2}}\right|^2.$$

The computation showing this is essentially the same as in the proof of Proposition 3.1, however one has to keep track of all the terms and the roles of V and $\tilde{V}$ are now reversed. On the other hand, we get for X_ζ that

$$\begin{aligned} X_\zeta &= (P - i\kappa - \tilde{V}(x))Y_\zeta \\ &= e^{i\langle x,\zeta\rangle}((P(i\zeta) - ik)y_{0,\zeta} - v_\zeta), \end{aligned}$$

where $v_\zeta \in L^2_\delta$. Since we assumed that the first component of $(P(\zeta) - ik)y_{0,\zeta}$ is zero, we also have

$$X^0_\zeta = -e^{i\langle x,\zeta\rangle}v^0_\zeta$$

and v^0_ζ satisfies

$$v^0_\zeta = -g_\zeta * (qv^0_\zeta),$$

and thus when $|\zeta| \to \infty$ the Sylvester–Uhlmann estimate implies that $v^0_\zeta = 0$ for $|\zeta|$ large enough. □

5 From inside to boundary

In this section we derive a formula that relates the material parameters inside M to the boundary values of the exponentially growing solution. The formula is related to the energy integral appearing in electrical impedance tomography, but due to the complexity of the full Maxwell system it is more involved.

To begin with, let $Y^*_0 \in \mathbf{\Omega}M$ be any solution of the homogenous space problem,

$$(P - ik)Y^*_0 = 0. \tag{44}$$

By using equation (40) and the decomposition of $\mathbf{\Delta}$, we have

$$\begin{aligned} (Y^*_0, QY) &= -(Y^*_0, (-\mathbf{\Delta} + k^2)Y) = -(Y^*_0, (P - ik)(P + ik)Y) \\ &= -(Y^*_0, (P - ik)\tilde{X}), \end{aligned} \tag{45}$$

we we denoted $\tilde{X} = (P+ik)Y$. By using Stokes formula (22) for P and equation (44), we find that

$$(Y^*_0, QY) = -\langle \mathbf{t}Y^*_0, \mathbf{n}X\rangle - \overline{\langle \mathbf{t}X, \mathbf{n}Y^*_0\rangle}.$$

Here, we used further the fact that at ∂M, $X = \tilde{X}$. Hence, if we know the boundary data $\{\mathbf{t}X, \mathbf{n}X\}$, we obtain an integral involving the potential Q over M.

To understand the significance of this relation better, let us look at the linearization of the left hand side of (45) with a particular choice of the form Y_0^*. The linearization means the approximation

$$(Y_0^*, QY) \approx (Y_0^*, QY_0). \tag{46}$$

Observe that in view of Theorem 4.2, this approximation is asymptotically valid as $|\zeta| \to \infty$. Following the original ideas of Calderón, let us choose $Y_0(x)$ as in the previous section. Similarly, we set

$$Y_0^*(x) = e^{i\langle x, \zeta^* \rangle} y_0^*.$$

where $\zeta^* \in \mathbb{C}^3$ is chosen such that for some fixed $\xi \in \mathbb{R}^3$, we have

$$\zeta + \zeta^* = \xi, \quad \langle \zeta^*, \zeta^* \rangle = k^2.$$

As we shall see in Section 7, in $\mathbb{C}^3$ there is enough space to make such a choice. The constant graded form y_0^* must be chosen again in such a way that equation (44) holds. it is easy to see that such a choice is obtained if we set, e.g.,

$$y_0^* = \frac{1}{|\zeta|}(P(i\zeta^*) + ik)z,$$

where $z = (z^0, z^1, z^2, z^3)$ is an arbitrary constant coefficient graded form and $P(i\zeta^*)$ is the symbol of the operator P, i.e.,

$$P(i\zeta^*) = e^{-i\langle x, \zeta^* \rangle} P e^{i\langle x, \zeta^* \rangle}.$$

To make this more explicit, one can introduce the Clifford–multiplication of forms by letting

$$fg = f \wedge g - \iota_f g$$

for a one–form f and a graded form g, and then extend this to all f in the obvious manner. Hence, if we identify a vector $\zeta = (\zeta_i)$ with the one–form $\zeta_i dx^i$, one sees that $P(\zeta)$ is just multiplication with ζ.

With these choices, we obtain

$$(Y_0, QY_0) = \int_M e^{i\langle x, \xi \rangle} (y_0^*, Q(x) y_0). \tag{47}$$

Hence, we see that within the linearization, the boundary values of X determine the Fourier transform of $(y_0^*, Q(x)y_0)$ and thus the function itself. In Section 7, we show how the material parameters $\mu(x)$ and $\gamma(x)$ can be recovered from this data.

6 From Λ to boundary values of X

In the previous section, we showed how the boundary values of X determine the integral (47). In this section, we show that the knowledge of the admittance map determines the boundary values of X.

The idea is to derive a version of the Stratton–Chu representation formula for the field X. To this end, we start with the Lippmann–Schwinger type equation for Y, and by writing

$$\tilde{X}(y) = (P - ik)Y(y),$$

we have

$$\begin{aligned} Y(x) &= Y_0(x) - \int_M \mathbf{G}(x-y) \wedge *Q(y)Y(y) \\ &= Y_0(x) + \int_M \mathbf{G}(x-y) \wedge *(-\mathbf{\Delta} + k^2)Y(y) \\ &= Y_0(x) + \int_M \mathbf{G}(x-y) \wedge *(P + ik)\tilde{X}(y). \end{aligned}$$

By writing $\mathbf{G}(x-y)$ in terms of the components of $\mathbf{I}$ as in (37), we obtain through integration by parts the equation

$$\begin{aligned} Y(x) &= Y_0(x) + \sum \omega_x^j \int_M G(x-y)\omega_y^j \wedge *(P + ik)\tilde{X}(y) \\ &= Y_0(x) + \sum \omega_x^j \int_M (-P_y + ik)G(x-y)\omega_y^j \wedge *\tilde{X}(y) \\ &\quad + \sum \omega_x^j \left(\int_{\partial M} \mathbf{t}G(x-y)\omega_y^j \wedge \mathbf{n}X(y) + \int_{\partial M} \mathbf{t}X(y) \wedge \mathbf{n}G(x-y)\omega_y^j \right). \end{aligned}$$

Here we used the fact that at the boundary, $\tilde{X}(y) = X(y)$ Assume that $x \in \mathbb{R}^3 \setminus M$, i.e., $\tilde{V}(x) = 0$ and $\kappa(x) = k$. We then have

$$X(x) = (P - ik)Y(x).$$

We substitute the integral representation of $Y(x)$ in this formula and use the fact that for $x \neq y$, we have

$$(P_x - ik)(-Py + ik)G(x-y)\mathbf{I} = 0,$$

and we arrive at the identity

$$\begin{aligned} X(x) &= X_0(x) + (P - ik)\sum \omega_x^j \left(\int_{\partial M} \mathbf{t}G(x-y)\omega_y^j \wedge \mathbf{n}X(y) \right. \\ &\qquad \left. + \int_{\partial M} \mathbf{t}X(y) \wedge \mathbf{n}G(x-y)\omega_y^j \right), \end{aligned}$$

where

$$X_0(x) = (P - ik)Y_0(x).$$

Assume now that $|\zeta|$ is large. Then, by Lemma 4.3, we have $X = (0, e, h, 0)$, and the boundary integral above takes the form

$$\begin{aligned} X(x) &= X_0(x) + (P - ik)\Big(\int_{\partial M} G(x-y)\mathbf{n}e, \sum_{j=1}^{3} dx^j \int_{\partial M} G(x-y)\mathbf{t}dy^j \wedge \mathbf{n}b, \\ &\sum_{j=1}^{3} \theta_j \int_{\partial M} G(x-y)\mathbf{t}e \wedge \mathbf{n}\nu_j, dV \int_{\partial M} G(x-y)\mathbf{t}h \Big). \end{aligned} \tag{48}$$

Letting the point x approach the boundary ∂M from the exterior domain $\mathbb{R}^3 \setminus M$ we obtain an integral equation for the boundary values of X. However, assuming that the impedance map Λ is known, it suffices to solve the tangential component of the electric field. Indeed, we have

$$\mathbf{n}b = \Lambda \mathbf{t}e,$$

and assuming that $|\zeta|$ is large, from Maxwell's equations

$$-\delta b + ike = 0, \quad de + ikb = 0,$$

we find that

$$\mathbf{n}e = \frac{1}{ik}\mathbf{n}\delta b = \frac{1}{ik}\mathbf{t}d * b.$$

Since the exterior derivative and the tangential trace commute, we have further

$$\mathbf{n}e = \frac{1}{ik}d_\partial \mathbf{n}b = \frac{1}{ik}d_\partial \Lambda \mathbf{t}e.$$

Here, d_∂ denotes the exterior derivative on ∂M. Similarly, we have

$$\mathbf{t}b = -\frac{1}{ik}\mathbf{t}de = -\frac{1}{ik}d_\partial \mathbf{t}e.$$

Summarizing, we have

$$\begin{aligned} \mathbf{t}X &= (0, \mathbf{t}e, \mathbf{t}b) = \left(0, \mathbf{t}e, -\frac{1}{ik}d_\partial \mathbf{t}e\right), \\ \mathbf{n}X &= (0, \mathbf{n}b, \mathbf{n}e) = \left(0, \Lambda\mathbf{t}e, \frac{1}{ik}d_\partial \Lambda \mathbf{t}e\right). \end{aligned}$$

Thus, we shall consider only the 1–form component of the system (48) and solve it for $\mathbf{t}e$. Denoting by e_0 the 1–form component of X_0 we have, for $x \in \mathbb{R}^3 \setminus M$,

$$\begin{aligned} e &= e_0 + d\int_{\partial M} G\mathbf{n}e - \delta \sum_{j=1}^{3} \theta_j \int_{\partial M} G\mathbf{t}e \wedge \mathbf{n}\nu_j - ik \sum_{j=1}^{3} dx^j \int_{\partial M} G\mathbf{t}dy^j \wedge \mathbf{n}b \\ &= e_0 + \frac{1}{ik}d\int_{\partial M} Gd\Lambda\mathbf{t}e - \delta \sum_{j=1}^{3} \theta_j \int_{\partial M} G\mathbf{t}e \wedge \mathbf{n}\nu_j - ik \sum_{j=1}^{3} dx^j \int_{\partial M} G\mathbf{t}dy^j \wedge \Lambda\mathbf{t}e. \end{aligned}$$

where the arguments of the functions are suppressed for brevity. Now we need to apply the tangential boundary trace from the exterior domain on both sides of this equation. To get a boundary integral equation, we need to take into account the jump relations of the layer potentials. Consider first the second integral on the right. By using the identites

$$\mathbf{n}\nu_j = \mathbf{t}dy^j, \quad \delta\theta_j f(x) = *df(x) \wedge dx^j,$$

we obtain

$$\delta \sum_{j=1}^{3} \theta_j \int_{\partial M} G\mathbf{t}e \wedge \mathbf{n}\nu_j = \sum_{j=1}^{3} \left(\int_{\partial M} \frac{\partial G}{\partial x^k} \mathbf{t}e \wedge dy^j \right) * (dx^k \wedge dx^j).$$

For simplicity, assume for a while that we use tangent–normal coordinates such that $M = \{x^3 \geq 0\}$. Then, $\mathbf{t}e \wedge dy^3 = 0$, and on the other hand, for $j = 1, 2$, we have

$$\mathbf{t}(*(dx^k \wedge dx^j)) = 0 \text{ unless } k = 3,$$

so finally

$$\begin{aligned}
& \mathbf{t} \sum_{j=1}^{3} \left(\int_{\partial M} \frac{\partial G}{\partial x^k} \mathbf{t}e \wedge dy^j \right) * (dx^k \wedge dx^j) \\
= \; & -\left(\int_{\partial M} \frac{\partial G}{\partial x^3} e_1 dy^1 \wedge dy^2 \right)\Big|_{\partial M}^{+} dx^1 - \left(\int_{\partial M} \frac{\partial G}{\partial x^3} e_2 dy^1 \wedge dy^2 \right)\Big|_{\partial M}^{+} dx^2 \\
= \; & \frac{1}{2}\mathbf{t}e - \sum_{j=1}^{3} \mathbf{n} \left(\int_{\partial M} d_x G\mathbf{t}e \wedge dy^j \right) \wedge dx^j,
\end{aligned}$$

the normal trace of the singular integral being understood in the sense of the principal value.

In the similar fashion we treat the first integral. Here we observe that since the tangential trace and the exterior derivative commute, the integral kernel has no derivatives of Green's function in the normal direction and hence the jump relations produce no extra terms besides the principal value integral.

By combining the terms, we reach the identity

$$\frac{1}{2}\mathbf{t}e = \mathbf{t}e_0 + D\Lambda\mathbf{t}e - K\mathbf{t}e, \tag{49}$$

where the operators D and K are given as

$$D\omega(x) = \frac{1}{ik}\left(d_\partial \mathbf{t} \int_{\partial M} G(x-y) d\omega(y) + k^2 \sum_{j=1}^{3} \int_{\partial M} G(x-y)\mathbf{t}dy_j \wedge \omega(y) \right),$$

$$K\omega(x) = \sum_{j=1}^{3} \mathbf{n} \left(\int_{\partial M} d_x G(x-y)\omega(y) \wedge dy^j \right) \wedge dx^j,$$

where $x \in \partial M$, $\omega \in \Omega^1 \partial M$ and the singular integrals being understood in the sense of principal values. Let us introduce the spaces

$$H(d, \Omega^k M) = \{f \in L^2(\Omega^k M); |\ df \in L^2(\Omega^{k+1} M)\},$$

$$H(\delta, \Omega^k M) = \{f \in L^2(\Omega^k M); |\ \delta f \in L^2(\Omega^{k-1} M)\}$$

and on the boundary

$$H^{-1/2}(d, \Omega^k \partial M) = \{g \in H^{-1/2}(\Omega^k \partial M); d_\partial g \in H^{-1/2}(\Omega^{k+1} \partial M)\}.$$

Then we have the bounded trace maps

$$\mathbf{t} : H(d, \Omega^k M) \to H^{-1/2}(d, \Omega^k \partial M)$$

and

$$\mathbf{n} : H(\delta, \Omega^k M) \to H^{-1/2}(d, \Omega^{3-k} \partial M)$$

and these maps are onto. Also, K maps $H^{-1/2}(d, \Omega^k \partial M)$ compactly to itself, and D just boundedly. For more details on this the reader is referred to [14]. Also there is large literature on layer potential techniques on (subdomains) of Riemnnian manifolds, even with Lipschitz–boundaries, see [12] and references therein. Of course, it is not known what is the analogue of the exponentially increasing Green's function in the general metric case.

It turns out that the equation (49) is of Fredholm type and has a unique solution exactly when ω is not an eigenfrequency for the interior Maxwell problem with vanishing tangential electric field. We shall not go in details here but refer to the (vector version) of this equation in the references [16] and [17].

7 From (Y_0^*, QY_0) to γ and μ

It turns out that for the reconstruction of γ and μ in the interior one doesn't need to recover the whole matrix Q, indeed since we don't know the relevant boundary data for the second order system, this cannot be done starting from the impedance map. However, as remarked in Section 5, one can still hope to extract information on Q. We start by making some explicit choices for the constant form Y_0. Fix $\xi \in \mathbb{R}^3$, and choose coordinates so that $\xi = (|\xi|, 0, 0)$. Then for $R > 0$ let

$$\zeta = \zeta(R) = (|\xi|/2, i(|\xi|^2/4 + R^2)^{1/2}, (R^2 + k^2)^{1/2}),$$

$$\zeta^* = \xi - \zeta.$$

Now $\langle \zeta, \zeta \rangle = \langle \zeta^*, \zeta^* \rangle = k^2$, and for some constant complex vectors a and b to be chosen later let (intepreted as a graded form)

$$y_{0,\zeta} = \frac{1}{|\zeta|}(\langle \zeta, a \rangle, kb, ka, \langle \zeta, b \rangle),$$

and for ζ^* let as before

$$y_{0,\zeta^*} = \frac{1}{|\zeta|}(P(i\zeta^*) + ik)z.$$

Then

$$(P + ik)Y_{0,\zeta^*} = (P + ik)Y_{0,\zeta} = 0$$

and as $|\zeta| \to \infty$ we have the limits

$$\lim y_{0,\zeta} = y_\infty = \langle \hat{\zeta}, a \rangle + \langle \hat{\zeta}. *_e b \rangle * (1),$$

$$\lim y_{0,\zeta^*} = y^*_\infty = -P(\hat{\zeta})z.$$

We are aiming for $(1,1)$ and $(8,8)$ components of the matrix Q, and to this end we choose first

$$(\hat{\zeta}, a) = 1, \quad (\hat{\zeta}, *_e b) = 0,$$

and let $z = a$ intepreted as a constant coefficient one–form, here $\hat{\zeta} = \zeta/|\zeta|$. Then

$$y^*_\infty = -1 - \hat{\zeta} \wedge a$$

and using the special form of the first row of Q we see that

$$\lim_{|\zeta| \to \infty} (Y^*_0, QY_\zeta) = \hat{Q}_{1,1}(\xi).$$

Similarly, reversing the roles of a and b above, and letting z be now a two form corresponding to b, we recover $\hat{Q}_{8,8}(\xi)$. On the other hand computing these terms and denoting $u = (\mu/\mu_0)^{1/2}$ and $v = (\gamma/\epsilon_0)^{1/2}$ we find that

$$Q_{1,1} = \delta\alpha + |\alpha|^2 - 4\omega^2(\gamma\mu - \epsilon_0\mu_0) = \frac{1}{v}(\Delta v - 4k^2 v(1 - uv))$$

and

$$Q_{8,8} = \delta\beta + |\beta|^2 - 4\omega^2(\gamma\mu - \epsilon_0\mu_0) = \frac{1}{u}(\Delta u - 4k^2 u(1 - uv)).$$

An application of the unique continuation principle for elliptic equations then shows that u and v, i.e. μ and γ are uniquely determined by the admittance map.

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