On Imaging Obstacles Inside Inhomogeneous Media*

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1 Introduction and outline of the method

In this paper we study the problem of recovering the shape of an obstacle inside a known inhomogeneous background. In particular, we study the problem arising from electrical impedance tomography (c.f. [K-S]) were one seeks a constructive method to determine the shape of an unknown obstacle inside an a priori known inhomogeneous anisotropic conducting medium by using current and voltage measurements on the boundary of some exterior domain surrounding the obstacle. We also study the inverse scattering problem of determining the shape of an obstacle inside a known inhomogeneous medium by using far field measurements. We model this by a Schrödinger equation with short range assumption for the potential. Since the energy is fixed in our case the methods and results work as well for the acoustic inverse scattering problem.

The first results in inverse obstacle scattering were based on Schiffer's proof by contradiction [L-S], or on relatively delicate high frequency asymptotics [Ma]. We refer to [C-K] for the history of the numerous advances on the problem. Nowadays, simple functional analysis arguments yield proofs which are constructive and only require scattering data at one fixed frequency. We will use a variant of the factorization method introduced by Andreas Kirsch. In a series of papers [Ki2,Ki2,Ki4] Kirsch presented an obstacle recovery method that is based on the following approach: Consider the measurement data to be an integral operator B that is defined on the measurement domain.The problem is to determine the shape of the obstacle being imaged from information contained in the operator B. The idea is to use a factorization for the operator B of the form

$$B = F\mathcal{S}F^*, \tag{1.1}$$

were S is an operator between function spaces defined on the boundary of the unknown domain and F is an operator with range in the measurement domain. From (1.1) one can prove that the knowledge of B determines the range of F which then determines the obstacle.

In this paper we solve the open problem of finding new factorizations which allow the method to be used in the case of inhomogeneous backgrounds. In addition, we extend the main functional analytic theorem of Kirsch (Theorem 2.3. in [Ki3]); our formulation leads to a computationally more efficient method of characterizing the obstacle.

We illustrate our approach on the following two inverse problems .

A. Electrical impedance tomography

Let Ω_1 and Ω_2 be bounded Lipschitz domains in $\mathbf{R}^n, n \geq 2$ with $\overline{\Omega_1} \subseteq \Omega_2$ and $\Omega_2 \setminus \overline{\Omega_1}$ connected. Let $\gamma(x) = (\gamma^{ij}(x))_{n\times n}$ be a known Lipschitz continuous real valued symmetric uniformly elliptic matrix function in Ω_2, which physically describes the anisotropic conductivity in Ω_2. Here Ω_1 represents the unknown obstacle lying inside Ω_2 and $\partial\Omega_2$ is the set where the measurements are made. We assume that on $\partial\Omega_1$ the unknown electric potential u satisfies the Dirichlet boundary condition. In Section 3 we study the conductivity equation

$$\sum_{i,j} \partial_i(\gamma^{ij}\partial_j u) = 0$$

for the electric potential u and introduce Dirichlet-to-Neumann maps Λ_1, Λ_2, Λ^{ij}, $i, j = 1, 2$. Then, if $\mathcal{S}_1$ is the single-layer operator, we will show, following [N2], the following

Theorem 1 *The factorization*

$$\Lambda^{22} - \Lambda_2 = \Lambda^{21}\mathcal{S}_1(\Lambda^{21})^*$$

holds as an operator $H^{\frac{1}{2}}(\partial\Omega_2) \to H^{-\frac{1}{2}}(\partial\Omega_2)$.

The operator $\Lambda^{22} - \Lambda_2$ ($= B$ in (1.1)) corresponds to the measurement data in $\partial\Omega_2$. We will show below that the factorization method gives a constructive way of uniquely determining the shape of the unknown obstacle Ω_1. In [B] the method was used to locate the support of inhomogeneities in a homogeneous background from the Neumann-to-Dirichlet map (see also [Ki4]). Here we seek the shape of an inpenatrable obstacle in an inhomogeneous object.

B. Inverse obstacle scattering in an inhomogeneous background

Let q be a real valued known background potential in $\mathbf{R}^n, n \geq 2$, such that

(1.2) $\quad q \in L^p_{loc}(\mathbf{R}^n)$, where $p > 2$, when $n = 2$ and $p = n$, when $n > 2$

and satisfying the short range decay condition

(1.3) $\quad q(x) = O(|x|^{-1-\varepsilon})$ as $|x| \to \infty$ for some $\varepsilon > 0$.

In Section 3 we study the Schrödinger equation

$$(-\Delta + q - k^2)u = 0$$

with fixed wave number $k > 0$ in $\mathbf{R}^n$ as well as a boundary value problem in $\mathbf{R}^n \setminus \overline{\Omega}$, where $\Omega \subseteq \mathbf{R}^n$ is the unknown obstacle satisfying either a Dirichlet or a Robin boundary condition on $\partial\Omega$. We prove the following factorization theorem. (The precise definitions of the operators can be found in Section 3.)

Theorem 2 *Let the potential q satisfy conditions (1.2) and (1.3) and let $\Omega \subseteq \mathbf{R}^n$ be a bounded Lipschitz domain with connected complement. The following factorizations hold:*

a)

$$(A_q - A^D_{\Omega,q})S^*_q = F^D \mathcal{S}^*_{k,q}(F^D)^*$$

and

$$(A_q - A^D_{\Omega,q})(S^D_{\Omega,q})^* = F^D \mathcal{S}_{k,q}(F^D)^*$$

in the Dirichlet case and

b)

$$(A_q - A^\sigma_{\Omega,q})S^*_q = F^\sigma (\mathcal{N}^\sigma_{k,q})^*(F^\sigma)^*$$

and

$$(A_q - A^\sigma_{\Omega,q})(S^\sigma_{\Omega,q})^* = F^\sigma \mathcal{N}^\sigma_{k,q}(F^\sigma)^*$$

in the Robin case.

We note that all the factorizations above are of the form (1.1) and that in the case when the inhomogeneity outside is known, the left hand sides of the equations in Theorem 2 can be determined from the far field measurements of the corresponding scattering problems.

Factorizations of the form

$$B = F_1 \mathcal{S}(F_2)^*$$

with different F_1 and F_2 corresponding to the case of a constant background medium first appeared in [N1] and for the case of a variable background in [I-N]. However for the factorization method to work we need F_1 to equal F_2 and in this case our factorizations appear to be new.

The uniqueness of the inverse problem of recovering the obstacle in af inhomogeneous background from the far field data was proved in [K-P]. The factorization method in this paper not only gives the uniqueness of the shape of the obstacle but also a reconstruction method for determining it. Furthermore, we are able to relax the smoothness assumption for the potential q from C^2 in [K-P] to L^p.

For the problem of determining the the shape of an obstacle by the factorization method in a constant backgound see [Ki2], [Ki3] and [G-K] and for the problem of determining the support of inhomogeneity from the far field data, see [Ki3,Ki4]. Note that the scattering problem in quantum mechanics can be easily transformed to one for acoustic scattering.

The factorization method presented here is based on the following functional analytic result, which extends Theorem 2.3 in [Ki3]. In [Ki3] it was assumed that in the equation (1.1) the operators F and B are compact and that the operator F is injective. The theorem below does not require these assumptions. We denote by H^* the dual of the Hilbert space H, by $(\, , \,)$ the inner product in H, linear in the first argument, and by $\langle \, , \, \rangle$ the duality pairing in $H \times H^*$.

Theorem 3 *Let H_1 and H_2 be Hilbert spaces. Suppose that $B = F\mathcal{S}F^*$ with $F : H_1 \to H_2$ bounded and $\mathcal{S} : H_1^* \to H_1$ bounded and coercive in the sense that there exist $c_1, c_2 > 0$ such that*

$$c_1\|g\|^2 \leq |\langle \mathcal{S}g, g\rangle| \leq c_2\|g\|^2 \ \ for \ all \quad g \in H_1^*. \tag{1.4}$$

Then for φ in H_2 the following conditions are equivalent:

(i) φ belongs to the range $\mathcal{R}(F)$ of F.

(ii) There exists $C < \infty$ such that

$$|\langle \varphi, \psi\rangle| \leq C|\langle B\psi, \psi\rangle|^{\frac{1}{2}} \ \ for \ all \quad \psi \in H_2^*. \tag{1.5}$$

(iii) φ is orthogonal to the subspace $\{\psi \ : \langle B\psi, \psi\rangle = 0\}$ and

$$\sup_{|\langle B\psi,\psi\rangle|=1} |\langle \varphi, \psi\rangle| < \infty. \tag{1.6}$$

Moreover, if any of these conditions is satisfied, then there exists $f \in H_1$ with $Ff = \varphi$ and

$$\sqrt{c_1} \sup_{|\langle B\psi,\psi\rangle|=1} |\langle \varphi, \psi\rangle| \leq \|f\| \leq \sqrt{c_2} \sup_{|\langle B\psi,\psi\rangle|=1} |\langle \varphi, \psi\rangle|. \tag{1.7}$$

In particular, the knowledge of B determines $\mathcal{R}(F)$.

In fact , we have the following more general consequence.

Corollary 4 *Let B and F be as in Theorem 3. Suppose that $A : H_2^* \to H_2$ is an operator that is comparable to B, i.e. there are constants d_1, d_2 such that for all ψ in H_2^**

$$d_1|\langle A\psi, \psi\rangle| \leq |\langle B\psi, \psi\rangle| \leq d_2|\langle A\psi, \psi\rangle|. \tag{1.8}$$

Then knowledge of A determines the range $\mathcal{R}(\mathcal{F})$ of F.

The proof and some alternative forms of the theorem are given in Section 2. We will see in Section 3 that the assumptions in Theorem 3 are valid for the impedance tomography problem (A) where $\mathcal{S} = \mathcal{S}_1$ and in the scattering problem (B) where $\mathcal{S}$ is either $\mathcal{S}_{k,q}$ or $\mathcal{N}_{k,q}^{\sigma}$ under the assumption that k^2 is neither a Dirichlet nor a Robin eigenvalue of $-\Delta + q$ in Ω. Note also that the condition (1.5) is equivalent to (2.11) in [Ki3]:

$$\inf\{|\langle B\psi, \psi\rangle| : \langle \varphi, \psi\rangle = 1\} > 0. \tag{1.9}$$

The following theorem allows us to recover the shape of an obstacle from knowledge of $\mathcal{R}(\mathcal{F})$. In the problem (A) we have $F = \Lambda^{21}$ and in (B) F is either F^D or F^{σ}. In Section 3 we define Green's functions $G_{\gamma}(\cdot,\cdot)$ in (A) and $G_{k,q}^{+}(\cdot,\cdot)$ with the far field pattern $(G_{k,q}^{+}(\cdot,\cdot))_{\infty}$ in (B). It is important that these Green's functions don't depend on the obstacles, and can be computed since the background inhomogeneity is known.

Theorem 5 *In problem (A) we have*

$$y \in \Omega_1 \text{ if and only if } \nu \cdot \gamma \nabla G_{\gamma}(\cdot, y)|_{\partial\Omega_2} \in \mathcal{R}(\Lambda^{21})$$

and in problem (B)

$$y \in \Omega \text{ if and only if } (G_{k,q}^{+}(\cdot, y))_{\infty} \in \mathcal{R}(F^{(D,\sigma)}).$$

The proof is given in Section 3.

Combining Theorems 1,2,3 and 5 (together with Proposition 3.6 which shows that the hypothesis of Theorem 3 is satisfied for problems (A) and (B)), we obtain our main results.

Theorem 6 *Let Ω_1 and Ω_2 be bounded Lipschitz domains in $\mathbb{R}^n, n \geq 2$, with $\overline{\Omega_1} \subset \Omega_2$ and $\Omega_2 \setminus \overline{\Omega_1}$ connected. Assume γ is a known Lipschitz continuous real-valued symmetric uniformly elliptic matrix function in Ω_2. Then the obstacle Ω_1 can be reconstructed from the knowledge of $\Lambda^{2,2}$ (the Cauchy data on $\partial\Omega_2$ for solutions of the conductivity equations in $\Omega_2 \setminus \overline{\Omega_1}$ with zero Dirichlet data on $\partial\Omega_1$).*

The corresponding inverse obstacle scattering result is the following

Theorem 7 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an unknown bounded Lipschitz domain with connected complement. Assume that the known potential q satisfies conditions (1.2)and (1.3) and that k^2 is not a Dirichlet or Robin eigenvalue of $-\Delta + q$ in Ω. Then Ω can be reconstructed from the knowledge of the scattering operator $S^D_{\Omega,q}$ (for the Dirichlet boundary condition) or $S^\sigma_{\Omega,q}$ (for the Robin condition) at the frequency k^2.*

Note that to recover Ω we do not need to know which operator, $S^D_{\Omega,q}$ or $S^\sigma_{\Omega,q}$, we were given, nor do we need to know σ.

Using Corollary 4 we obtain the following surprising uniqueness, and, in some sense, stability result which we formulate, for simplicity, for vanishing q (i.e. homogeneous background).

Theorem 8 *Let $q = 0$. If the scattering operators for two bounded Lipschitz obstacles are comparable (in the sense of (1.8)) at one non-resonant frequency k^2, then the obstacles must coincide.*

Below we list, for convenience, the steps of the reconstruction algorithm.

1. Choose sampling points y_j in Ω_2 (problem (A)) or in $\mathbf{R}^n$ (problem(B)) and evaluate the functions $\nu \cdot \gamma \nabla G_\gamma(\cdot, y_j)|_{\partial\Omega_2}$ or the far field patterns $(G^+_{k,q}(\cdot, y_j))_\infty$ for the corresponding Green's functions; we denote these functions by φ_j. Note that φ_j can be calculated without any knowledge of the obstacle. Note also that, in problem(B), if the potential q is zero in$\mathbb{R}^n$, then we have $(G^+_{k,q}(\cdot, y))_\infty(\hat{x}) = e^{-ik\hat{x}\cdot y}$

2. Using the measurement data B determine the subspace $\{\psi : < B\psi, \psi >= 0\}$ and the 'unit ball' $\{\psi \in H_2^* : |\langle B\psi, \psi\rangle| = 1\}$. This is done only once without any reference to the sampling points.

3. For each sampling point y_j check whether φ_j is orthogonal to $\{\psi : < B\psi, \psi >= 0\}$ and whether

$$\sup_{|\langle B\psi,\psi\rangle|=1} |\langle \varphi_j, \psi \rangle|$$

   is finite or not. By Theorems 3 and 5 we can then conclude whether or not y_j lies inside the obstacle.

Note that if B happens to be compact and normal and $\{\psi_j\}$ is an orthonormal basis of eigenvectors corresponding to eigenvalues $\{\lambda_j\}$, then $\psi = \sum_j a_j\psi_j$ satisfies

$$|\langle B\psi, \psi\rangle| = 1$$

if and only if

$$|\sum_j \lambda_j |a_j|^2| = 1.$$

Furthermore in this case

$$\sup_{|\langle B\psi,\psi\rangle|=1} |\langle\varphi, \psi\rangle|$$

is comparable to $(\sum_j \frac{|\langle\varphi,\psi_j\rangle|}{|\lambda_j|})^{\frac{1}{2}}$, and we recover the obstacle identification criterion in [Ki1].

2 Functional analytic results for factorizations

In this section we prove Theorem 3 and also give some alternative versions and consequences. We denote by $\kappa_i : H_i \to H_i^*, i = 1, 2$, the isometric linear isomorphism defined by

$$(2.1) \qquad (f, g)_{H_i} = \langle f, \kappa_i g\rangle \quad for\ all \quad f, g \in H_i, i = 1, 2.$$

We start with a simple general lemma, which gives a good starting point for understanding Theorem 3. For completeness we provide a proof here.

Lemma 2.1 *Let $F : H_1 \to H_2$ be a bounded operator and let $\varphi \in H_2$. Then $\varphi \in \mathcal{R}(F)$ if and only if there exist $C < \infty$ such that for all $\psi \in H_2^*$*

$$(2.2) \qquad |\langle\varphi, \psi\rangle| \leq C\|F^*\psi\|_{H_1^*}.$$

On $\mathcal{R}(F)$ the operator F has a right inverse $J : \mathcal{R}(F) \to H_1$ such that for all $\varphi \in \mathcal{R}(F)$

$$(2.3) \qquad \|J\varphi\|_{H_1} = \sup_{\|F^*\psi\|=1} |\langle\varphi, \psi\rangle|.$$

Proof. If $\varphi = Ff$ for some $f \in H_1$, then $for\ all \quad \psi \in H_2^*$ we have

$$|\langle\varphi, \psi\rangle| = |\langle Ff, \psi\rangle| = |\langle f, F^*\psi\rangle| \leq \|f\|\|F^*\psi\|.$$

Assume now that $\varphi \in H_2$ and (2.2) holds. Let us define the linear functional j_φ on $\mathcal{R}(F^*)$ by

$$j_\varphi(F^*\psi) = \langle \varphi, \psi \rangle.$$

By (2.2), if $F^*\psi_1 = F^*\psi_2$, then $\langle \varphi, \psi_1 - \psi_2 \rangle = 0$ and therefore j_φ is well-defined. Moreover, $\|j_\varphi\| = C_\varphi$, where C_φ is the smallest constant for which (2.2) holds. We can extend j_φ uniquely to a bounded linear functional on $\overline{\mathcal{R}(F^*)}$ and then to all of H_1^* by setting $j_\varphi \equiv 0$ on $\overline{\mathcal{R}(F^*)}^\perp$. By the reflexivity there exists a unique $f_\varphi \in H_1$ with $j_\varphi(g) = \langle f_\varphi, g \rangle$ and $\|f_\varphi\|_{H_1} = C_\varphi$. We define $J\varphi := f_\varphi$, so that

$$\langle FJ\varphi, \psi \rangle = \langle f_\varphi, F^*\psi \rangle = j_\varphi(F^*\psi) = \langle \varphi, \psi \rangle$$

for all $\psi \in H_2^*$. So $FJ\varphi = \varphi$ and then (2.3) follows from

$$\|J\varphi\| = C_\varphi = \sup_{\|F^*\psi\|=1} |\langle \varphi, \psi \rangle|.$$

□

Note that one can consider Lemma 2.1 as a special case of Theorem 3 when $\mathcal{S}$ is κ_1^{-1}. Next, to obtain Therem 3 from Lemma 2.1, we observe that condition (1.4) in fact requires $\mathcal{S}$ to be comparable to κ_1^{-1}.

Proof of Theorem 3. If $\varphi = Ff$ then, in view of (1.4), we have for all ψ in H_2^*

$$|\langle \varphi, \psi \rangle| \le \|f\| \|F^*\psi\| \overset{(1.4)}{\le} \frac{\|f\|}{\sqrt{c_1}} |\langle \mathcal{S}F^*\psi, F^*\psi \rangle|^{\frac{1}{2}} = \frac{\|f\|}{\sqrt{c_1}} |\langle B\psi, \psi \rangle|^{\frac{1}{2}}. \tag{2.4}$$

This shows that (i) implies (ii). If ϕ satisfies condition (ii) then

$$|\langle \varphi, \psi \rangle| \le C |\langle B\psi, \psi \rangle|^{\frac{1}{2}} = C |\langle \mathcal{S}F^*\psi, F^*\psi \rangle|^{\frac{1}{2}} \le C\sqrt{c_2} \|F^*\psi\|.$$

Thus by Lemma 2.1 there exists $f \in H_1$ such that $\varphi = Ff$ and

$$\|f\|_{H_1} = \sup_{\|F^*\psi\|=1} |\langle \varphi, \psi \rangle| = \sup_{F^*\psi \neq 0} \frac{|\langle \varphi, \psi \rangle|}{\|F^*\psi\|}. \tag{2.5}$$

This shows that (ii) implies (i). To see that (ii) also implies (1.7) we use hypothesis (1.4) which yields, as above,

$$(2.6) \qquad \frac{1}{\sqrt{c_2}}|\langle B\psi,\psi\rangle|^{\frac{1}{2}} \le \|F^*(\psi)\|_{H_1} \le \frac{1}{\sqrt{c_1}}|\langle B\psi,\psi\rangle|^{\frac{1}{2}}$$

for any ψ in H_2^*. (In particular$F^*\psi = 0$ if and only if $\langle B\psi,\psi\rangle = 0$). Substituting (2.6) to (2.5) we obtain

$$(2.7) \qquad \sqrt{c_1}\sup_{\langle B\psi,\psi\rangle\neq 0}\frac{|\langle\varphi,\psi\rangle|}{|\langle B\psi,\psi\rangle|^{\frac{1}{2}}} \le \|f\|_{H_1} \le \sqrt{c_2}\sup_{\langle B\psi,\psi\rangle\neq 0}\frac{|\langle\varphi,\psi\rangle|}{|\langle B\psi,\psi\rangle|^{\frac{1}{2}}}$$

and(1.7) follows. Moreover if ψ is such that $\langle B\psi,\psi\rangle = 0$ (equivalently, if ψ is in the null-space of F^*) then(1.5) implies $\langle\varphi,\psi\rangle = 0$. We have thus shown that (ii) implies (iii).

Finally we verify that (iii) implies (ii). Assume φ satisfies (iii) with

$$(2.8) \qquad C_\phi = \sup_{|\langle B\psi,\psi\rangle|=1}|\langle\varphi,|\psi\rangle|.$$

Let $\psi \in H_2^*$. Then either $\langle B\psi,\psi\rangle = 0$, in which case $\langle\varphi,\psi\rangle = 0$, so (1.5) holds, or $\langle B\psi,\psi\rangle \neq 0$. In the latter case, we let $\tilde{\psi} = \psi/|\langle B\psi,\psi\rangle|^{1/2}$, so that $|\langle B\tilde{\psi},\tilde{\psi}\rangle| = 1$. Then $|\langle\varphi,\tilde{\psi}\rangle| \le C_\varphi$, i.e. $|\langle\varphi,\psi\rangle| \le C_\varphi|\langle B\psi,\psi\rangle|^{1/2}$, so that (1.5) holds with $C = C_\varphi$.

□

We use the isomorphism κ_2 to identify H_2 and H_2^* and define operator $\tilde{B} := \kappa_2 B : H_2^* \to H_2^*$ so that we have for all $f,g \in H_2^*$

$$\langle Bf,g\rangle = (\tilde{B}f,g)_{H_2^*}.$$

With the usual definitions of $|\tilde{B}|$ and $|\tilde{B}|^{1/2}$ by the formula $|\tilde{B}| = (\tilde{B}^*\tilde{B})^{1/2}$ we can thus use κ_2 to define $|B|$ and $|B|^{1/2}$.

Theorem 2.2 *Let the conditions of Theorem 3 be valid and assume that, in addition B satisfies*

$$(2.9) \qquad c_3\langle|B|\psi,\psi\rangle \le |\langle B\psi,\psi\rangle| \le c_4\langle|B|\psi,\psi\rangle \;, c_3, c_4 > 0$$

under the previous interpretations. Then (1.5) $\Leftrightarrow$ $\varphi \in \mathcal{R}(|B|^{\frac{1}{2}})$. If B is also injective, then $C_\varphi \sim \|\, |B|^{-1/2}\phi \,\|$; more precisely,

$$(2.10) \qquad \frac{1}{\sqrt{c_4}}\|\,|B|^{-\frac{1}{2}}\varphi\| \le \sup_{|\langle B\psi,\psi\rangle|=1}|\langle\varphi,\psi\rangle| \le \frac{1}{\sqrt{c_3}}\|\,|B|^{-\frac{1}{2}}\varphi\|.$$

In particular when $\mathcal{S}$ is bounded and coercive and $B = F\mathcal{S}F^*$ satisfies (2.9), then $\mathcal{R}(F) = \mathcal{R}(|B|^{\frac{1}{2}})$.

Proof. Assume that B satisfies (2.9). Then the condition (1.5) is equivalent to

$$|\langle \varphi, \psi \rangle| \le C |\langle |B|\psi, \psi \rangle|^{\frac{1}{2}} = C \| |B|^{\frac{1}{2}} \psi \| \; for \; all \quad \psi \in H_2^*$$

which is equivalent to $\varphi \in \mathcal{R}(|B|^{\frac{1}{2}})$ by Lemma 2.1.

By (2.9) we can write

$$\frac{1}{\sqrt{c_4}} \sup_{\| |B|^{\frac{1}{2}} \psi \| = 1} |\langle \varphi, \psi \rangle| = \frac{1}{\sqrt{c_4}} \sup \frac{|\langle \varphi, \psi \rangle|}{\langle |B|\psi, \psi \rangle^{\frac{1}{2}}} \le \sup \frac{|\langle \varphi, \psi \rangle|}{|\langle B\psi, \psi \rangle|^{\frac{1}{2}}}$$

$$= \sup_{|\langle B\psi, \psi \rangle| = 1} |\langle \varphi, \psi \rangle| \le \frac{1}{\sqrt{c_3}} \sup \frac{|\langle \varphi, \psi \rangle|}{\langle |B|\psi, \psi \rangle^{\frac{1}{2}}} = \frac{1}{\sqrt{c_3}} \sup_{\| |B|^{\frac{1}{2}} \psi \| = 1} |\langle \varphi, \psi \rangle|.$$

Assuming now that B is injective, then also $|B|^{\frac{1}{2}}$ is injective and has a dense range so that

$$\sup_{\| |B|^{\frac{1}{2}} \psi \| = 1} |\langle \varphi, \psi \rangle| = \sup_{\substack{\|u\|=1 \\ u \in \mathcal{R}(|B|^{\frac{1}{2}})}} |\langle \varphi, |B|^{-\frac{1}{2}} u \rangle|$$

$$= \sup_{\substack{\|u\|=1 \\ u \in \mathcal{R}(|B|^{\frac{1}{2}})}} |\langle |B|^{-\frac{1}{2}} \varphi, u \rangle| = \| |B|^{-\frac{1}{2}} \varphi \|$$

which proves (2.10).

□

Remarks. 1) Note that in Theorem 2.2 we can assume instead of (2.9) that there exists a positive operator $\bar{B}$ s.t.

$$c_3 \langle \bar{B}\psi, \psi \rangle \le |\langle B\psi, \psi \rangle| \le c_4 \langle \bar{B}\psi, \psi \rangle \; for \; all \quad \psi \in H_2^*.$$

Then (1.5) $\Leftrightarrow \varphi \in \mathcal{R}(\bar{B}^{\frac{1}{2}})$ and if $\bar{B}$ is also injective, then

$$\sup_{|\langle B\psi, \psi \rangle| = 1} |\langle \varphi, \psi \rangle| \approx \| \bar{B}^{-\frac{1}{2}} \varphi \|.$$

2) If instead of (1.4) in Theorem 3 we assume that

$$c_1 \| \mathcal{S}^* g \|^2 \le |\langle \mathcal{S}^* g, g \rangle| \le c_2 \| \mathcal{S}^* g \|^2 \; for \; all \quad g \in H_1^*$$

then by an easy modification of the proof we can see that $\varphi \in \mathcal{R}(F\mathcal{S})$ if and only if there exist $C < \infty$ such that $|\langle \varphi, \psi \rangle| \le C |\langle B\psi, \psi \rangle|^{\frac{1}{2}}$ for all $\psi \in H_2^*$.

3 Factorizations for recovering obstacles in an inhomogeneous backgrounds

In this section we obtain the factorization theorems 1 and 2 after we have introduced the relevant operators. We also show that the conditions in Theorem 3 are valid in both special cases (A) and (B).

A. Electric impedance tomography

Let Ω_1 and Ω_2 be bounded Lipschitz domains in $\mathbf{R}^n, n \geq 2$, with $\overline{\Omega_1} \subseteq \Omega_2$ and $\Omega_2 \setminus \overline{\Omega_1}$ connected. Let

$$\gamma(x) = (\gamma^{ij}(x))_{n\times n} \in C^{0,1}(\Omega_2)$$

be a known Lipschitz continuous real valued symmetric matrix function, which is uniformly elliptic, i.e. there exist constants $c, C > 0$ s.t.

$$c|\xi|^2 \leq \sum_{i,j} \gamma^{ij}(x)\xi_i\xi_j \leq C|\xi|^2 \; for \; all \quad \xi \in \mathbf{R}^n, x \in \Omega_2. \tag{3.1}$$

Physically γ describes the anisotropic conductivity in Ω_2. Following [N2] we define six Dirichlet-to-Neumann maps in the following way:

Let $u \in H^1(\Omega_2)$ be the weak solution of the conductivity equation

$$\nabla \cdot (\gamma \nabla u) = 0 \text{ in } \Omega_2, \tag{3.2}$$

or in coordinate form

$$\sum_{i,j} \partial_i(\gamma^{ij}\partial_j u) = 0 \text{ in } \Omega_2,$$

satisfying the Dirichlet boundary condition

$$u|_{\partial\Omega_2} = f \in H^{\frac{1}{2}}(\partial\Omega_2). \tag{3.3}$$

Let $\nu(x)$ denote the outer unit normal on the boundary of Ω_2. We assume that all function spaces in this section have real scalar fields. Define

$$\Lambda_2 f = \nu \cdot \gamma \nabla u|_{\partial\Omega_2} \in H^{-\frac{1}{2}}(\partial\Omega_2) \tag{3.4}$$

in the weak sense, i.e.

$$(3.5)\qquad \int_{\Omega_2} \sum_{i,j} \gamma^{ij}\partial_i u\, \partial_j v\, dx = \langle \tau_2 v, \Lambda_2 f\rangle \quad \textit{for all} \quad v \in H^1(\Omega_2),$$

where $\tau_2 : H^1(\Omega_2) \to H^{\frac{1}{2}}(\partial\Omega_2)$ is the trace-mapping and $\langle\cdot,\cdot\rangle$ is the bilinear duality pairing in $H^{\frac{1}{2}}(\partial\Omega_2) \times H^{-\frac{1}{2}}(\partial\Omega_2)$.

Physically the conductivity equation (3.2) corresponds to the absence of sources and sinks of the electric current generated by the potential with given boundary values (3.3). The Dirichlet-to-Neumann map (3.4) then gives the current flux across the boundary $\partial\Omega_2$ in the static situation.

Let $\Lambda_1 : H^{\frac{1}{2}}(\partial\Omega_1) \to H^{-\frac{1}{2}}(\partial\Omega_1)$ be the corresponding map with Ω_1 in place of Ω_2. On the boundary $\partial\Omega_1$ the direction of $\nu(x)$ is chosen to be outwards from Ω_1.

For $j = 1$ or 2, let $u_j \in H^1(\Omega_2 \setminus \overline{\Omega_1})$ be the weak solutions of

$$(3.6)\qquad \begin{cases} \nabla\cdot(\gamma\nabla u_j) &= 0 & \text{in } \Omega_2\setminus\overline{\Omega_1} \\ u_j &= f \in H^{\frac{1}{2}}(\partial\Omega_j) & \text{on } \partial\Omega_j \\ u_j &= 0 & \text{on the rest of the boundary.} \end{cases}$$

We define

$$(3.7)\qquad \Lambda^{1j} f := \nu\cdot\gamma\nabla u_j|_{\partial\Omega_1}$$

and

$$(3.8)\qquad \Lambda^{2j} f := \nu\cdot\gamma\nabla u_j|_{\partial\Omega_2}.$$

Note that the Dirichlet-to-Neumann map in $\Omega_2 \setminus \overline{\Omega_1}$ can be written as a matrix

$$(3.9)\qquad \begin{pmatrix} -\Lambda^{11} & -\Lambda^{12} \\ \Lambda^{21} & \Lambda^{22} \end{pmatrix} : H^{\frac{1}{2}}(\partial\Omega_1) \oplus H^{\frac{1}{2}}(\partial\Omega_2) \to H^{-\frac{1}{2}}(\partial\Omega_1) \oplus H^{-\frac{1}{2}}(\partial\Omega_2)$$

if the outward normal is chosen on the boundary of the domain as usual. This formula and the symmetry of the D-to-N map imply that

$$(3.10)\qquad (\Lambda^{11})^* = \Lambda^{11}, (\Lambda^{22})^* = \Lambda^{22} \text{ and } (\Lambda^{12})^* = -\Lambda^{21}.$$

Let $\mathcal{G}_\gamma$ be the inverse operator of the isomorphism

$$-\sum_{i,j} \partial_i \gamma^{ij} \partial_j : H_0^1(\Omega_2) \to H^{-1}(\Omega_2)$$

and define the self-adjoint single-layer operator

$$\mathcal{S}_1 = \tau_1 \mathcal{G}_\gamma \tau_1^* : H^{-\frac{1}{2}}(\partial\Omega_1) \to H^{\frac{1}{2}}(\partial\Omega_1), \tag{3.11}$$

where $\tau_1 : H_0^1(\Omega_2) \to H^{\frac{1}{2}}(\partial\Omega_1)$ is the trace-mapping. The integral kernel $G_\gamma(x,y)$ of $\mathcal{G}_\gamma : H^{-1}(\Omega_2) \to H_0^1(\Omega_2)$ is the Dirichlet Green's function satisfying

$$\begin{cases} \nabla_x \cdot \gamma \nabla_x G_\gamma(x,y) &= -\delta_y(x) \quad \text{in } \Omega_2 \\ G_\gamma(x,y) &= 0 \qquad\quad \text{for } x \in \partial\Omega_2, y \in \Omega_2 \end{cases} \tag{3.12}$$

and we can also write

$$\mathcal{S}_1 g(x) := \int\limits_{\partial\Omega_1} G_\gamma(x,y) g(y)\, d\sigma(y)\ ,\ x \in \partial\Omega_1. \tag{3.13}$$

It is a well-known consequence of the classical de Giorgi-Nash theorem that $G_\gamma(x,y)$ is continuous in $\{(x,y) \in \Omega_2 \times \Omega_2 : x \neq y\}$. The singularity of $G_\gamma(\cdot,y)$ at y is given by (see [Se]):

$$G_\gamma(x,y) \approx \begin{cases} |x-y|^{2-n} & ,n > 2 \\ \log \frac{1}{|x-y|} & ,n = 2 \end{cases} \tag{3.14}$$

when x is in a neighborhood of y. Here $f \approx g$ means that $\exists c_1, c_2 > 0$ s.t. $c_1\, g \le f \le c_2\, g$.

One can easily modify the results of Section 6 in [N2] to see that the following factorization is true:

$$\Lambda^{22} - \Lambda_2 = -\Lambda^{21} \mathcal{S}_1 \Lambda^{12} = \Lambda^{21} \mathcal{S}_1 (\Lambda^{21})^*, \tag{3.15}$$

which shows the validity of (1.1) when we choose

$$\begin{cases} H_1 = H^{\frac{1}{2}}(\partial\Omega_1)\,,\ H_2 = H^{-\frac{1}{2}}(\partial\Omega_2), \\ F = \Lambda^{21} \text{ and } B = \Lambda^{22} - \Lambda_2. \end{cases}$$

The inverse problem in this case can now be formulated: determine the shape of the unknown scatterer Ω_1 from the knowledge of B, which is interpreted as the given data: $\Lambda^{2,2}$ is measured, and Λ_2 can be calculated (since Ω_2 and γ are known).

B. Scattering in an inhomogeneous background

We denote by H_0 the self-adjoint extension of the Laplacian $-\Delta$ in $L^2(\mathbf{R}^n), n \geq 2$, with domain $\mathcal{D}(H_0) = H^2(\mathbf{R}^n)$. It is well known that the spectrum of

H_0 consists entirely of the absolutely continuous spectrum and $\sigma(H_0) = \sigma_{(ac)}(H_0) = [0, \infty)$.

Now consider the perturbed differential operator $H_0 + q$, where q is assumed to be a real valued potential satisfying the previous conditions (1.2) and (1.3) in Section 1. By Hölder's inequality one can then see that

$$\tilde{q}(x) := (1 + |x|)^{1+\varepsilon'} q(x) \ , \ 0 < \varepsilon' < \varepsilon,$$

belongs to the Stummel class

$$M_\rho := \{f : \mathbf{R}^n \to \mathbf{R} : \sup_{x \in \mathbf{R}^n} \int\limits_{|x-y| \leq 1} |f(y)|^2 |x-y|^{\rho-n} \, dy < \infty, \text{ when } \rho < n$$

$$\text{or } \sup_{x \in \mathbf{R}^n} \int\limits_{|x-y| \leq 1} |f(y)|^2 \, dy =: \sup_{x \in \mathbf{R}^n} N_f(x) < \infty \text{ when } \rho \geq n\}$$

for some $\rho < 4$ and that also $N_{\tilde{q}}(x) \to 0$ as $|x| \to \infty$. These conditions are sufficient for the multiplication operator $\tilde{q}$ to be H_0-compact (i.e. $\tilde{q} : H^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ is compact) (see e.g. [S]). It follows that q is compact operator from $H^2_s(\mathbf{R}^n)$ into $L^2_{s+1+\varepsilon}(\mathbf{R}^n)$ for sufficiently small $\varepsilon > 0$ and for every $s \in \mathbf{R}$. Here

$$L^2_s(\mathbf{R}^n) = \{u(x) : (1 + |x|^2)^{s/2} u(x) \in L^2(\mathbf{R}^n)\} \tag{3.16}$$

with norm

$$\|u\|_{0,s} = \|(1 + |x|^2)^{s/2} u\|_{L^2(\mathbf{R}^n)} \tag{3.17}$$

for every integer $m \geq 0$

$$H^m_s(\mathbf{R}^n) = \{u(x) : D^\alpha u \in L^2_s(\mathbf{R}^n), \ 0 \leq |\alpha| \leq m\} \tag{3.18}$$

with the weighted Sobolev norm

$$\|u\|_{m,s} = (\sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{0,s})^{\frac{1}{2}}. \tag{3.19}$$

The H_0-bound for q is 0, so the Kato-Rellich theorem [R-S II] shows that

$$H := H_0 + q$$

is self-adjoint and that $\mathcal{D}(H) = \mathcal{D}(H_0) = H^2(\mathbf{R}^n)$. Weyl's theorem [R-S IV] implies that the essential spectrum $\sigma_{(ess)}(H)$ (by definition the union of accumulation points of $\sigma(H)$ and isolated eigenvalues of infinite multiplicity) equals the essential spectrum of H_0, which is $[0, \infty)$, so that the negative part of $\sigma(H)$ may consist only of denumerable set of eigenvalues with finite multiplicity and the only possible accumulation point for them is 0.

Agmon [A] considered the larger class of potentials $q \in L^2_{loc}(\mathbf{R}^n)$ such that the multiplication operator $(1 + |x|)^{1+\varepsilon} q(x)$ is H_0-compact for some $\varepsilon > 0$. In that case there might be imbedded eigenvalues on the positive real axis, but as a consequence of the unique continuation principle it is known that with our assumptions for q that cannot happen [J-K].

For a fixed $k > 0$ we denote by $\Phi^{\pm}_k(x - y)$ the integral kernel of the resolvent $R^{\pm}_0(k) = (H_0 - k^2 \mp i0)^{-1}$, which exists as a bounded operator from L^2_s into H^2_{-s} for all $s > \frac{1}{2}$ [A]. It is well known that

$$\Phi^{\pm}_k(x) = \frac{i}{4}\left(\frac{k}{2\pi|x|}\right)^{\frac{n-2}{2}} H^{(1,2)}_{\frac{n-2}{2}}(k|x|) \quad \left(= \frac{e^{\pm ik|x|}}{4\pi|x|}, \text{ when } n = 3\right), \tag{3.20}$$

where $H^{(1,2)}_p$ are the Hankel functions of the first and second kind. Here the $+$ stands for the outgoing and $-$ for the incoming fundamental solution.

Now we define the radiation condition as a modification of the definition in [A] which allows us to consider exterior boundary value problems.

Definition 3.1 A function $u \in L^2_{loc}(\mathbf{R}^n)$ is $(k-)$outgoing if

$$u = R^{+}_0(k) f \tag{3.21}$$

outside some compact set for some $f \in L^2_s(\mathbf{R}^n)$, $s > \frac{1}{2}$. If $R^+_0(k)$ is replaced by $R^-_0(k)$ we say that u is $(k-)$incoming.

Note that this definition allows us to change u inside some compact set without disturbing the radiation condition, in particular it allows us to leave u undefined in any compact set.

The asymptotics of $u = R^+_0(k) f$ is known to be

$$u(x) = C_n \frac{k^{\frac{n-3}{2}} e^{ik|x|}}{|x|^{\frac{n-1}{2}}} \widehat{f}(k\omega) + u_0(x), \tag{3.22}$$

where

$$C_n = \frac{1}{4\pi}\left(\frac{1}{2\pi i}\right)^{\frac{n-3}{2}}, \tag{3.23}$$

$\omega = \frac{x}{|x|} \in S^{n-1}$ and

$$\lim_{r\to\infty} r^{-1} \int\limits_{|x|\le r} |u_0(x)|^2\, dx = 0. \tag{3.24}$$

Here the Fourier transform is defined by

$$\widehat{f}(\xi) = \int\limits_{\mathbf{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

We can now define the far field pattern for outgoing functions:

Definition 3.2 If $u = R_0^+(k)f, f \in L_s^2, s > 1/2$, is an outgoing function, we define the far field pattern $u_\infty \in L^2(S^{n-1})$ of u by

$$u_\infty(\omega) = \widehat{f}(k\omega) = \int\limits_{\mathbf{R}^n} e^{-ik\omega\cdot x} f(x)\, dx \quad \text{for } \omega \in S^{n-1}. \tag{3.25}$$

The far field pattern is well defined because the Fourier transform maps L_s^2 into H^s, so when $s > 1/2$ we can apply the trace theorem on the sphere kS^{n-1}. Also the far field pattern does not depend on f: let f_1 and f_2 be such that outside some compact set K we have $u = R_0^+(k)f_1 = R_0^+(k)f_2$. Then

$$R_0^+(k)(f_1 - f_2) =: w \in H^2_{comp}$$

and so

$$f_1 - f_2 = (H_0 - k^2)w$$

which implies that $\widehat{f_1}|_{kS^{n-1}} = \widehat{f_2}|_{kS^{n-1}}$.

The next step is to define scattering solutions to the Schrödinger equation

$$(H - k^2)\varphi = 0, \quad k > 0 \text{ fixed}, \tag{3.26}$$

in the form

$$\varphi^\pm(x,\theta) = e^{ikx\cdot\theta} + \varphi_s^\pm(x,\theta)\ , \quad \theta \in S^{n-1}, \tag{3.27}$$

where $\varphi_s^\pm$ is outgoing/incoming. If the potential q is compactly supported φ^+ is defined as the solution of the Lippmann-Schwinger equation

$$\varphi^\pm(x,\theta) = e^{ikx\cdot\theta} - \int\limits_{\mathbf{R}^n} \Phi_k^\pm(x-y)q(y)\varphi^\pm(y,\theta)\, dy; \tag{3.28}$$

$\varphi_s^{\pm}$ solves the integral equation

$$(3.29)\quad \varphi_s^{\pm}(x,\theta) + \int_{\mathbf{R}^n} \Phi_k^{\pm}(x-y)q(y)\varphi_s^{\pm}(y)\,dy = -\int_{\mathbf{R}^n} \Phi_k^{\pm}(x-y)q(y)e^{ik\theta\cdot y}\,dy.$$

Since $R_0^{\pm}(k)q : H^2_{-s} \to H^2_{-s}$ is compact for any $s > \frac{1}{2}$, Fredholm theory yields a unique solution of (3.29) in H^2_{-s}. The uniqueness follows from the classical Rellich's lemma and the unique continuation principle (UCP) [J-K]:

Theorem 3.3 (Unique continuation principle) *Let $\Omega \subseteq \mathbf{R}^n$ be open and connected. If $n \geq 3$, let $V \in L^{n/2}_{loc}(\Omega), f \in H^{2,p}_{loc}(\Omega), p = \frac{2n}{n+2}$ and $|\Delta f(x)| \leq |V(x)||f(x)|$ a.e. in Ω. If f vanishes in a neighborhood of some $x_0 \in \Omega$, then $f \equiv 0$ in Ω. When $n = 2$, the conclusion holds if $V \in L^p_{loc}(\Omega)$ for some $p > 1$ and $f \in H^{2,1}_{loc}(\Omega)$.*

In order to allow the more general short range decay assumption on the potential q we use instead of the plane wave $e^{ikx\cdot\theta}$ the Herglotz wave operator [C-K] , [R-S II]

$$(3.30)\qquad L^2(S^{n-1}) \to H^2_{-s}(\mathbf{R}^n) : g \mapsto \int_{S^{n-1}} e^{ikx\cdot\theta} g(\theta)\,d\sigma(\theta)$$

and interpret the solution function $\varphi^{\pm}(x,\theta)$ as a bounded operator

$$(3.31)\qquad L^2(S^{n-1}) \to H^2_{-s}(\mathbf{R}^n) : g \mapsto \int_{S^{n-1}} \varphi^{\pm}(x,\theta) g(\theta)\,d\sigma(\theta)$$

for $s > 1/2$ (see also [I-N]). We denote the right hand side of (3.30) and (3.31) by φ_g^0 and $\varphi_g^{\pm}$ respectively. The boundedness of the Herglotz wave operator follows easily from duality and the trace theorem. Now we define the scattering solution $\varphi_g^{\pm}$ for $g \in L^2(S^{n-1})$ as the solution of

$$(3.32)\qquad \varphi_g^{\pm} = \varphi_g^0 - R_0^{\pm}(k)(q\varphi_g^{\pm})$$

or

$$(3.33)\qquad \varphi_g^{\pm} = (I + R_0^{\pm}(k)q)^{-1}\varphi_g^0.$$

Here $I + R_0^{\pm}(k)q$ is indeed invertible as a bounded operator in H^2_{-s} for $s > 1/2$ [A]. Note that in the case of compactly supported q the equation (3.32) follows from (3.28) by multiplying by $g(\theta)$ and by integrating with respect to θ.

By using the resolvent

$$R_q^{\pm}(k) = (H - k^2 \mp i0)^{-1} : L_s^2 \to H_{-s}^2 \, , s > \frac{1}{2}$$

[A] (note the absence of positive eigenvalues) we can also write

$$\varphi_g^{\pm} = (I - R_q^{\pm}(k)q)(\varphi_g^0). \tag{3.34}$$

We denote by $G_{k,q}^{\pm}$ the radiating Green's function for the scattering problem, i.e. the integral kernel of $R_q^{\pm}(k)$. The solution of the Lippmann-Schwinger equation (3.28) can then be written as

$$\varphi^{\pm}(x,\theta) = e^{ikx\cdot\theta} - \int_{\mathbf{R}^n} G_{k,q}^{\pm}(x,y)q(y)e^{iky\cdot\theta}\,dy \tag{3.35}$$

corresponding to equation (3.34).

We now collect some well known properties of the scattering solution φ^+ when q is bounded and compactly supported, see e.g. [C-K]. The corresponding formulas for φ^- are also valid. The function φ^+ has the asymptotic behaviour

$$\varphi^+(x,\theta) = e^{ikx\cdot\theta} + C_n \frac{k^{\frac{n-3}{2}} e^{ik|x|}}{|x|^{\frac{n-1}{2}}} A_q(\omega,\theta) + o(\frac{1}{|x|^{\frac{n-1}{2}}}),$$

where $\omega = \frac{x}{|x|} \in S^{n-1}$ and C_n was defined by (3.23). Here the scattering amplitude $A_q(\omega,\theta)$ has the representation as an integral

$$A_q(\omega,\theta) = -\int_{\mathbf{R}^n} e^{-ik\hat{x}\cdot y} q(y)\varphi^+(y,\theta)\,dy \tag{3.36}$$

which exists as a continuous function.

We denote the corresponding integral operator (the far field operator) by A_q

$$A_q f(\omega) = \int_{S^{n-1}} A_q(\omega,\theta) f(\theta)\,d\sigma(\theta),$$

which is bounded and compact as a map $L^2(S^{n-1}) \to L^2(S^{n-1})$ and define the scattering operator

$$S_q = I + \frac{i}{4\pi}(\frac{k}{2\pi})^{n-2} A_q : L^2(S^{n-1}) \to L^2(S^{n-1}). \tag{3.37}$$

It is well known that S_q is unitary.

We now turn to the general q and extend the definitions of A_q and S_q to this case. We define the far field operator $A_q \in B(L^2(S^{n-1}), L^2(S^{n-1}))$ by

$$A_q g(\omega) = (\varphi_g^{\pm} - \varphi_g^0)_\infty(\omega) = -\widehat{(q\varphi_g^{\pm})}(k\omega). \tag{3.38}$$

Since the multiplication by q is a compact operator from H_s^2 into $L_{s+1+\varepsilon}^2$ for any $s < -1/2$ and for sufficiently small $\varepsilon > 0$ we see easily that A_q is compact. The formula (3.36) shows that this definition indeed extends the earlier one for compactly supported potentials. The scattering operator S_q is defined now also by the formula (3.37). The unitarity of S_q can be proved (see [A]) by approximating the potential q by the sequence of bounded compactly supported potentials

$$q_j(x) = \begin{cases} q(x) & \text{when } |x| \le j \text{ and } |q(x)| \le j \\ 0 & \text{otherwise} \end{cases} \tag{3.39}$$

and using the unitarity of S_{q_j} for every $j \ge 1$.

Now we turn to the obstacle case. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded Lipschitz domain such that the complement $\mathbf{R}^n \setminus \overline{\Omega}$ is connected. Physically Ω is interpreted as an unknown scatterer whose shape is to be found by using far field measurements.

Consider the following exterior boundary value problem: find the outgoing solution u to the Schrödinger equation

$$(-\Delta + q - k^2)u = 0 \quad \text{in } \mathbf{R}^n \setminus \overline{\Omega} \tag{3.40}$$

where the boundary condition is either the Dirichlet condition

$$u|_{\partial\Omega} = f \in H^{\frac{1}{2}}(\partial\Omega) \tag{3.41}$$

or the Robin condition

$$(\frac{\partial}{\partial\nu}u + \sigma u)|_{\partial\Omega} = f \in H^{-\frac{1}{2}}(\partial\Omega); \tag{3.42}$$

here $\sigma \in L^\infty(\partial\Omega)$ is a given real valued function. The Neumann boundary condition is considered as a special case $\sigma \equiv 0$.

Theorem 3.4 *The problem (3.40) and either (3.41) or (3.42) has a unique outgoing solution $u \in H_{loc}^2(\mathbf{R}^n \setminus \overline{\Omega}) \cap H^1(\widetilde{\Omega} \setminus \overline{\Omega})$ where $\widetilde{\Omega}$ is any smooth bounded open set containing $\overline{\Omega}$.*

Proof. We first verify uniqueness. Let u be the solution of the problem satisfying either $u \equiv 0$ or $\frac{\partial}{\partial \nu} u + \sigma u \equiv 0$ on the boundary $\partial\Omega$. Because u is outgoing, we have $u = R_0^+(k)g$ outside some compact set K ($\supseteq \Omega$) for some $g \in L_s^2(\mathbf{R}^n)$. We may also take K so large that q is bounded outside K. We will show that

$$Im\langle R_0^+ g, g\rangle = 0, \tag{3.43}$$

which implies that $u_\infty = \widehat{g}|_{kS^{n-1}} = 0$, see [A]. Then the asymptotics (3.22) and the fact that u satisfies Schrödinger equation outside K imply that there exists a sequence $(r_n), r_n \to \infty$ so that

$$\lim_{n\to\infty} \int\limits_{|x|=r_n} |\partial_r u|^2 + |u|^2 \, d\sigma(x) = 0$$

(see [G-Y]). The classical theorem of Kato [K] forces u to vanish outside K. The unique continuation principle then shows that u must be equal to zero in $\mathbf{R}^n \setminus \overline{\Omega}$, which proves the uniqueness part of the theorem. Now let us prove (3.43). Let $R > 0$ be so large that $K \subseteq B_R = \{x : |x| < R\}$ and let $v = R_0^+ g \in H_{-s}^2$. Green's first identity implies that

$$\int\limits_{|x|\le R} v\overline{g}\, dx = \int\limits_{|x|\le R} v\overline{(H_0 - k^2)v}\, dx$$

$$= \int\limits_{|x|\le R} |\nabla v|^2 - k^2|v|^2\, dx - \int\limits_{|x|=R} v\overline{\frac{\partial v}{\partial \nu}}\, d\sigma.$$

Taking the imaginary part, letting $R \to \infty$ and noting that Green's identity applied to u in $B_R \setminus \overline{\Omega}$ implies

$$Im \int\limits_{|x|=R} v\overline{\frac{\partial v}{\partial \nu}}\, d\sigma = Im \int\limits_{|x|=R} u\overline{\frac{\partial u}{\partial \nu}}\, d\sigma = Im \int\limits_{\partial\Omega} u\overline{\frac{\partial u}{\partial \nu}}\, d\sigma = 0,$$

we obtain (3.43).

Now let us prove the existence part. At first we choose a smooth bounded open set $\widetilde{\Omega}$ containing $\overline{\Omega}$ s.t. k^2 is neither a Dirichlet nor a Robin eigenvalue for $-\Delta + q$ either in $\widetilde{\Omega} \setminus \overline{\Omega}$ or in $\widetilde{\Omega}$. This can always be done by modifying $\widetilde{\Omega}$ if needed. Let's fix the cut-off function $\chi \in C^\infty(\mathbf{R}^n)$ s.t. $\chi \equiv 1$ in the neighbourhood of $\overline{\Omega}$ and supp $\chi \subseteq \widetilde{\Omega}$. We denote by Ω' the set $\widetilde{\Omega} \setminus \overline{\Omega}$.

The idea of the proof is to seek the solution u in the form

$$u = R_q^+(k)\rho + \chi v \tag{3.44}$$

where $\rho \in L^2(\Omega')$ is solved by the Fredholm theory and $v \in H^1(\Omega') \cap H^2_{loc}(\Omega')$ is chosen in an appropriate way to take care of the boundary behaviour. Note that such u is outgoing because of the resolvent equation

$$R_q^+(k) = R_0^+ - R_0^+ q R_q^+. \tag{3.45}$$

We start with the Dirichlet case. The substitution of (3.44) into the equation (3.40) leads to

$$\rho + [\chi(-\Delta + q - k^2) - \Delta\chi]v - 2\nabla\chi \cdot \nabla v = 0. \tag{3.46}$$

We now write $v = v_f - v_\rho$ where $v_f \in H^1(\Omega')$ and $v_\rho \in H^1(\Omega')$ satisfy the equations

$$(-\Delta + q - k^2)v_f = (-\Delta + q - k^2)v_\rho = 0 \text{ in } \Omega' \tag{3.47}$$

such that

$$v_f|_{\partial\Omega} = f \quad \text{and} \quad v_f|_{\partial\widetilde{\Omega}} = 0 \tag{3.48}$$

and

$$v_\rho|_{\partial\Omega} = R_q^+(k)\rho|_{\partial\Omega} \quad \text{and} \quad v_\rho|_{\partial\widetilde{\Omega}} = 0. \tag{3.49}$$

Here ρ belongs to $L^2(\Omega')$; also note that v_f can be solved without ρ. The standard interior regularity theorem implies that v_f and v_ρ belong to $H^2_{loc}(\Omega')$ and so does u. Note that Hölder's inequality and Sobolev's embedding theorem imply that qv_f and $qv_\rho \in L^2_{loc}(\Omega')$.

We can write equation (3.46) in the form

$$\rho + K\rho = (\Delta\chi)v_f + 2\nabla\chi \cdot \nabla v_f \in L^2(\Omega'), \tag{3.50}$$

where K is the compact operator in $L^2(\Omega')$ taking $\rho \in L^2(\Omega')$ first to $v_\rho \in H^1(\Omega')$ and then to $(\Delta\chi)v_\rho + 2\nabla\chi \cdot \nabla v_\rho \in L^2(\Omega')$.

We will now show that the operator $I + K$ is injective and then the Fredholm alternative completes the proof.

Let $(I + K)\rho = 0$, where $\rho \in L^2(\Omega')$. Now (3.44) and (3.46) imply that the function

$$\tilde{u} = R_q^+(k)\rho - \chi v_\rho$$

is an outgoing solution of (3.40) satisfying $\tilde{u}|_{\partial\Omega} = 0$. Then by the uniqueness we must have $\tilde{u} = 0$ outside Ω or

$$R_q^+(k)\rho = \chi v_\rho \quad \text{in } \mathbf{R}^n \setminus \overline{\Omega}.$$

Since $\chi \equiv 1$ is the neighbourhood of $\partial\Omega$, the definition of K implies that $K\rho$ and then also ρ is equivalently 0 near $\partial\Omega$. But this means that we can extend v_ρ into Ω as $R_q^+(k)\rho$ obtaining the solution of

$$(-\Delta + q - k^2)v_\rho = 0 \quad \text{in } \widetilde{\Omega}$$

which is 0 on the boundary $\partial\widetilde{\Omega}$. But then v_ρ has to vanish by our assumption for $\widetilde{\Omega}$ and then also ρ must be 0. The Dirichlet case is now proved. The Robin case is handled similarly; all we have to do is to use the corresponding boundary condition in defining v.

□

We now define the far field operators for Dirichlet and Robin boundary value problems, $F^D : H^{1/2}(\partial\Omega) \to L^2(S^{n-1})$ and $F^\sigma : H^{-1/2}(\partial\Omega) \to L^2(S^{n-1})$, by setting $F^D f$ and $F^\sigma f$ to be far field patterns of the outgoing solutions of the corresponding exterior boundary value problems (3.40) and (3.41) or (3.42) with data f.

Let us consider the following obstacle scattering problem

$$(3.51) \qquad \begin{cases} (-\Delta + q - k^2)\, u = 0 \text{ in } \mathbf{R}^n \setminus \overline{\Omega} \\ u(x,\theta) = e^{ik\theta\cdot x} + u_s(x,\theta) \\ u_s(\cdot,\theta) \text{ is outgoing} \\ u|_{\partial\Omega} = 0 \text{ or } (\frac{\partial}{\partial\nu}u + \sigma u)|_{\partial\Omega} = 0. \end{cases}$$

To allow for potentials q which satisfy the short range decaying condition, we use Herglotz operators as before. We formulate the problem in the operator form:

For $g \in L^2(S^{n-1})$ find u_g such that $u_g - \varphi_g^0$ is outgoing,

$$(3.52) \qquad (-\Delta + q - k^2)(u_g - \varphi_q^0) = -q\varphi_q^0 \quad \text{in } \mathbf{R}^n \setminus \overline{\Omega}$$

and either

$$(3.53) \qquad u_g|_{\partial\Omega} = 0$$

or

$$(3.54) \qquad (\frac{\partial}{\partial\nu}u_g + \sigma u_g)|_{\partial\Omega} = 0.$$

Theorem 3.5 *The problem (3.52) and either (3.53) or (3.54) has a unique solution* $u_g \in H^2_{loc}(\mathbf{R}^n \setminus \overline{\Omega}) \cap H^1(\widetilde{\Omega} \setminus \overline{\Omega})$ *where* $\widetilde{\Omega}$ *is any smooth bounded open set containing* $\overline{\Omega}$.

Proof. The proof is similar to the proof of Theorem 3.4 except that we seek the solution in the form

$$u_g - \varphi_g^0 = R_q^+(k)(-q\varphi_g^0) + R_q^+(k)\rho + \chi v \tag{3.55}$$

instead of (3.44) to take care of the right-hand side of (3.52).

□

We now define the obstacle far field operators

$$A^D_{\Omega,q} \text{ and } A^\sigma_{\Omega,q} \in B(L^2(S^{n-1}), L^2(S^{n-1}))$$

and corresponding unitary scattering operators by using Theorem 3.5. Here the superscript D refers to the Dirichlet case and σ to the Robin case. $A^{(D,\sigma)}_{\Omega,q}$ are defined by the sequence of operators

$$g \mapsto \varphi_g^0 \mapsto u_g \mapsto (u_g - \varphi_g^0)_\infty \tag{3.56}$$

and they are clearly compact. $S^{(D,\sigma)}_{\Omega,q} \in B(L^2(S^{n-1}), L^2(S^{n-1}))$ are then defined by

$$S^{(D,\sigma)}_{\Omega,q} = I + \frac{i}{4\pi}(\frac{k}{2\pi})^{n-2} A^{(D,\sigma)}_{\Omega,q}. \tag{3.57}$$

We note here that because of the continuous dependence of $R_q^+(k)$ on q [A], that is

$$\lim_{j\to\infty} R_j^+(k) = R_q^+(k) \ \text{ in } B(L^2_s, H^2_{-s}) \tag{3.58}$$

where $R_j^+(k)$ corresponds to the bounded compactly supported potential q_j defined by the formula (3.39), as a consequence of Theorems 3.4 and 3.5 the operators $A^{(D,\sigma)}_{\Omega,q}$ and $S^{(D,\sigma)}_{\Omega,q}$ are limits of the corresponding operators for q_j. The same applies also for operators F^D and F^σ. For compactly supported potential q_j we note that the corresponding u_{q_j} satisfies Helmholtz equation outside some compact set, and we see that the operators $S^{(D,\sigma)}_{\Omega,q_j}$ are unitary by modifying the proof in [Ki1]. The unitarity of $S^{(D,\sigma)}_{\Omega,q}$ then follows by the above limiting procedure.

Next we fefine layer potentials and layer operators. For proofs of the following results in Lipschitz domains, see [C], [M] or [I-N]. Let $\widetilde{\Omega}$ be a smooth open bounded set containing $\partial\Omega$. By duality and by interpolation we see that the operator $R_q^+(k)$ is bounded from $H_s^t(\mathbf{R}^n)$ into $H_{-s}^{t+2}(\mathbf{R}^n)$ for $t \in [-2, 0]$ and $s > 1/2$. We define the single layer potential

$$SL\, f := R_q^+(k)\tau^* f \quad \text{for } f \in H^{-\frac{1}{2}}(\partial\Omega), \tag{3.59}$$

where τ is a trace mapping $H^1(\widetilde{\Omega}) \to H^{\frac{1}{2}}(\partial\Omega)$, and the single layer operator

$$\mathcal{S}_{k,q} f := \tau(SL\, f|_{\widetilde{\Omega}}). \tag{3.60}$$

The definitions are obviously independent of the choice of $\widetilde{\Omega}$ and we see that $\mathcal{S}_{k,q}$ maps $H^{-\frac{1}{2}}(\partial\Omega)$ continuously into $H^{\frac{1}{2}}(\partial\Omega)$. By using the integral kernel $G_{k,q}^+(x, y)$ we can also write

$$\mathcal{S}_{k,q} f(x) = \int_{\partial\Omega} G_{k,q}^+(x, y) f(y)\, d\sigma(y) \quad , \; x \in \partial\Omega.$$

The following jump relations are valid

$$(\tau^+ - \tau^-) SL\, f = 0 \;\text{ and }\; \Big(\frac{\partial}{\partial\nu}^+ - \frac{\partial}{\partial\nu}^-\Big) SL\, f = -f, \tag{3.61}$$

where $\tau^\pm$ and $\frac{\partial}{\partial\nu}^\pm$ are trace and normal derivative operators on $\partial\Omega$ from $\widetilde{\Omega} \setminus \overline{\Omega}$ $(+)$ and $\widetilde{\Omega} \cap \Omega$ $(-)$.

The double layer potential is defined by

$$DL\, f := R_q^+(k)\widetilde{f} \in L_{-s}^2 \quad \text{for } f \in H^{\frac{1}{2}}(\partial\Omega), s > 1/2 \tag{3.62}$$

where

$$\langle \widetilde{f}, \varphi \rangle = \int_{\partial\Omega} f \overline{\frac{\partial\varphi}{\partial\nu}}\, d\sigma \quad \text{for } \varphi \in H^2(\widetilde{\Omega}).$$

It can be shown ([M] , [I-N]), that actually $DL\, f|_{\widetilde{\Omega}} \in H^1(\widetilde{\Omega})$ and that the jump relations

$$(\tau^+ - \tau^-) DL\, f = f \;\text{ and }\; \Big(\frac{\partial}{\partial\nu}^+ - \frac{\partial}{\partial\nu}^-\Big) DL\, f = 0 \tag{3.63}$$

are valid. Here $\frac{\partial}{\partial\nu}^\pm DL$ are bounded from $H^{\frac{1}{2}}(\partial\Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$. We need the Robin modification of the normal derivative of double layer potential $\mathcal{N}_{k,q}^\sigma$:

$$\mathcal{N}_{k,q}^\sigma f := \Big(\frac{\partial}{\partial\nu}^+ + \sigma\tau^+\Big)|_{\partial\Omega}(DL\, f + SL\, \sigma f). \tag{3.64}$$

It is a consequence of the previous jump relations that we also have

$$\mathcal{N}^{\sigma}_{k,q} f = (\frac{\partial}{\partial\nu}^{-} + \sigma\tau^{-})|_{\partial\Omega}(DL\, f + SL\, \sigma f) \tag{3.65}$$

and that $\mathcal{N}^{\sigma}_{k,q}$ is bounded from $H^{\frac{1}{2}}(\partial\Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$. We also write

$$\mathcal{N}^{\sigma}_{k,q} f(x) = (\frac{\partial}{\partial\nu(x)} + \sigma(x)) \int_{\partial\Omega} (\frac{\partial}{\partial\nu(y)} + \sigma(y)) G^{+}_{k,q}(x,y) f(y)\, d\sigma(y) \ , \ x \in \partial\Omega.$$

We are now ready to prove Theorem 2.

Proof of Theorem 2. We already saw that all of the operators in the claims depend continuously on q in the sense we formulated before. Therefore it is enough to prove the theorem only for potentials that are bounded and compactly supported. Let us assume this in the rest of the proof.

The Green's formula and the well known asymptotics

$$G^{+}_{k,q}(x,y) = C_n \frac{k^{\frac{n-3}{2}} e^{ik|y|}}{|y|^{\frac{n-1}{2}}} \varphi^{+}(x, -\hat{y}) + o(\frac{1}{|y|^{\frac{n-1}{2}}}) \tag{3.66}$$

lead to the identity

$$G^{+}_{k,q}(x,y) - G^{-}_{k,q}(x,y) = \frac{i}{4\pi}(\frac{k}{2\pi})^{n-2} \int_{S^{n-1}} \varphi^{+}(y, \hat{z}) \varphi^{-}(x, -\hat{z})\, d\sigma(\hat{z}). \tag{3.67}$$

Subtracting the two equations (3.35) and using (3.67) and (3.36) we get

$$\varphi^{+}(x,\theta) = \varphi^{-}(x,\theta) - \frac{i}{4\pi}(\frac{k}{2\pi})^{n-2} \int_{\mathbf{R}^n} \int_{S^{n-1}} \varphi^{+}(y,\hat{z})\varphi^{-}(x,-\hat{z})\, d\sigma(\hat{z}) \cdot q(y) e^{iky\cdot\theta}\, dy$$

$$= \varphi^{-}(x,\theta) + \frac{i}{4\pi}(\frac{k}{2\pi})^{n-2} \int_{S^{n-1}} A_q(-\theta, \hat{z}) \varphi^{-}(x, -\hat{z})\, d\sigma(\hat{z}) \tag{3.68}$$

$$= RS_q R\varphi^{-}(x, \cdot)(\theta), \tag{3.69}$$

where $Rf(\theta) = f(-\theta), \theta \in S^{n-1}$.

We denote the restriction of the Herglotz operator (3.31) by $H_q : L^2(S^{n-1}) \to H^{\frac{1}{2}}(\partial\Omega)$ so that

$$H_q g(x) := \int_{S^{n-1}} \varphi^{+}(x,\alpha) g(\alpha)\, ds(\alpha) \ , \ x \in \partial\Omega.$$

Its adjoint $H_q^*: H^{-\frac{1}{2}}(\partial\Omega) \to L^2(S^{n-1})$ satisfies

$$H_q^* f(\alpha) = \int\limits_{\partial\Omega} \varphi^-(x,-\alpha) f(x)\, ds(x)\ ,\ \alpha \in S^{n-1},$$

where the identity $\overline{\varphi^+(x,\theta)} = \varphi^-(x,-\theta)$ was used. Multiplying (3.69) by $f(x)$ and integrating over $\partial\Omega$, we get

$$\int\limits_{\partial\Omega} \varphi^+(x,-\theta) f(x)\, ds(x) = S_q H_q^* f(\theta). \tag{3.70}$$

If $u(x,\theta)$ is the solution of (3.51) with the Dirichlet boundary condition, then $\varphi^+(x,\theta) - u(x,\theta) = (\varphi^+(x,\theta) - e^{ik\theta\cdot x}) - (u(x,\theta) - e^{ik\theta\cdot x})$ is the outgoing solution for the Schrödinger equation in the exterior domain $\mathbf{R}^n \setminus \overline{\Omega}$ whose restiriction to $\partial\Omega$ is $\varphi^+(\cdot,\theta)|_{\partial\Omega}$. From this we conclude that

$$F^D(\varphi^+(\cdot,\alpha)|_{\partial\Omega})(\theta) = A_q(\theta,\alpha) - A^D_{\Omega,q}(\theta,\alpha). \tag{3.71}$$

Green's function's asymptotics (3.66) shows that the left hand side of equation (3.70) equals $F^D \mathcal{S}_{k,q} f(\theta)$ so using the unitarity of S_q we get

$$H_q^* = S_q^* F^D \mathcal{S}_{k,q} \text{ or } H_q = \mathcal{S}_{k,q}^* (F^D)^* S_q,$$

which together with equation (3.71) gives

$$F^D \mathcal{S}_{k,q}^* (F^D)^* S_q = F^D H_q = A_q - A^D_{\Omega,q}$$

or

$$F^D \mathcal{S}_{k,q}^* (F^D)^* = (A_q - A^D_{\Omega,q}) S_q^*.$$

Denoting

$$\rho = \frac{1}{4\pi}(\frac{k}{2\pi})^{n-2}$$

and by taking the adjoints we have

$$\begin{aligned}
F^D \mathcal{S}_{k,q} (F^D)^* &= S_q(A_q^* - A_{\Omega,q}^*) = (I + \frac{i}{4\pi}(\frac{k}{2\pi})^{n-2} A_q)(A_q^* - A_{\Omega,q}^*) \\
&= A_q^* - A_{\Omega,q}^* + i\rho A_q A_q^* - i\rho A_q A_{\Omega,q}^* \\
&= A_q - i\rho A_q A_q^* - A_{\Omega,q} + i\rho A_{\Omega,q} A_{\Omega,q}^* + i\rho A_q A_q^* - i\rho A_q A_{\Omega,q}^* \\
&= (A_q - A_{\Omega,q})(I - i\rho A_{\Omega,q}^*) = (A_q - A_{\Omega,q}) S_{\Omega,q}^*,
\end{aligned}$$

because the unitarity of $S = I + i\rho A$ implies that $A - A^* = i\rho A A^*$. The proof of the part a) is now complete.

To prove part b) we define the operator $H_q^\sigma : L^2(S^{n-1}) \to H^{-\frac{1}{2}}(\partial\Omega)$ by

$$H_q^\sigma g(x) := \int_{S^{n-1}} (\frac{\partial}{\partial\nu(x)} + \sigma(x))\varphi^+(x,\alpha)g(\alpha)\, d\sigma(\alpha)\ ,\ x \in \partial\Omega,$$

so that $(H_q^\sigma)^* : H^{\frac{1}{2}}(\partial\Omega) \to L^2(S^{n-1})$ has the representation

$$(H_q^\sigma)^* g(\alpha) = \int_{\partial\Omega} [(\frac{\partial}{\partial\nu(x)} + \sigma(x))\varphi^-(x,-\alpha)]g(x)\, d\sigma(x)\ ,\ \alpha \in S^{n-1}.$$

Differentiating (3.68) with respect to x, adding the same equation multiplied by $\sigma(x)$, multipying by $f(x)$ and integrating over $\partial\Omega$ we get

$$\int_{\partial\Omega} [(\frac{\partial}{\partial\nu(x)} + \sigma(x))\varphi^+(x,-\theta)]f(x)\, d\sigma(x)$$

$$= \int_{\partial\Omega} [(\frac{\partial}{\partial\nu(x)} + \sigma(x))\varphi^-(x,-\theta)]f(x)\, d\sigma(x)$$

$$+\frac{i}{4\pi}(\frac{k}{2\pi})^{n-2} \int_{S^{n-1}} \int_{\partial\Omega} A_q(\theta,\hat{z})[(\frac{\partial}{\partial\nu(x)} + \sigma(x))\varphi^-(x,-\hat{z})]f(x)\, d\sigma(\hat{z})\, d\sigma(x)$$

$$= S_q(H_q^\sigma)^* f(\theta).$$

Because of the asymptotics (3.66) the left side of this equation equals $F^\sigma \mathcal{N}_{k,q}^\sigma f(\theta)$, so we have

$$H_q^\sigma = (\mathcal{N}_{k,q}^\sigma)^*(F^\sigma)^* S_q.$$

Similarly to part a) by using the solution $u(x,\theta)$ of the Robin boundary value problem (3.51) we can now write

$$F^\sigma((\frac{\partial}{\partial\nu} + \sigma)\varphi^+(\cdot,\alpha)|_{\partial\Omega})(\theta) = A_q(\theta,\alpha) - A_{\Omega,q}^\sigma(\theta,\alpha),$$

so as operators we obtain

$$(A_q - A_{\Omega,q}^\sigma) = F^\sigma H_q^\sigma = F^\sigma(\mathcal{N}_{k,q}^\sigma)^*(F^\sigma)^* S_q.$$

The second formula of part b) is established in the exactly same way as in part a), so the theorem is now proved.

Q.E.D

In particular Theorem 3 makes it possible to determine $\mathcal{R}(F)$ from the knowledge of operator B. We now prove Theorem 5, which shows that the unknown object in problem (A) and (B) can be recovered from $\mathcal{R}(\mathcal{F})$.

Proof of Theorem 5. We first consider case (A). Let $y \in \Omega_1$. Since $G_\gamma(\cdot, y)$ is the H^1 solution of the conductivity equation (3.2) in $\Omega_2 \setminus \overline{\Omega_1}$ satisfying $G_\gamma(\cdot, y)|_{\partial\Omega_1} \in H^{\frac{1}{2}}(\partial\Omega_1)$ and $G_\gamma(\cdot, y)|_{\partial\Omega_2} = 0$, we have

$$\nu \cdot \gamma \nabla G_\gamma(\cdot, y)|_{\partial\Omega_2} = \Lambda^{21}(G_\gamma(\cdot, y)|_{\partial\Omega_1}).$$

Next let $y \in \Omega_2 \setminus \overline{\Omega_1}$ and assume that there exists $f \in H^{\frac{1}{2}}(\partial\Omega_1)$ s.t. $\nu \cdot \gamma \nabla G_\gamma(\cdot, y)|_{\partial\Omega_2} = \Lambda^{21} f$. Let $u \in H^1(\Omega_2 \setminus \overline{\Omega_1})$ be the corresponding solution of the equation (3.2), $u|_{\partial\Omega_1} = f$ and $u|_{\partial\Omega_2} = 0$. Let $r > 0$ be so small that $B_r(y) \subseteq \Omega_2$, in which case $v := G_\gamma(\cdot, y) - u$ satisfies (3.2) in $\Omega_2 \setminus (\overline{\Omega_1} \cup \overline{B_r(y)})$ with vanishing Cauchy data on $\partial\Omega_2$. Extend γ as an arbitrary Lipschitz continuous function to the ball $B_R(0)$ where $R > 0$ is chosen so that $\overline{\Omega_2} \subseteq B_R(0)$. The zero function extension $v_0 \in H^1(B_R(0) \setminus (\overline{\Omega_1} \cup \overline{B_r(y)}))$ of v is a solution of (3.2) as one can see by applying Green's formula to any test function $\varphi \in C_0^\infty(B_R(0) \setminus (\overline{\Omega_1} \cup \overline{B_r(y)}))$ and using $\nu \cdot \gamma \nabla v_0 = 0$ on $\partial\Omega_2$. The unique continuation principle [H2] implies that v vanishes and therefore $G_\gamma(\cdot, y) = u$ in $\Omega_2 \setminus (\overline{\Omega_1} \cup \overline{B_r(y)})$. By letting $r \to 0$ we reach a contradiction because u is continuous in $\Omega_2 \setminus \overline{\Omega_1}$ but $G_\gamma(\cdot, y)$ is singular (by (3.14)).

Finally, when $y \in \partial\Omega_1$ the assumption that there exists $f \in H^{\frac{1}{2}}(\partial\Omega_1)$ so that $\nu \cdot \gamma \nabla G_\gamma(\cdot, y)|_{\partial\Omega_2} = \Lambda^{21} f$ implies by the similar reasoning that $G_\gamma(\cdot, y)|_{\Omega_2 \setminus \overline{\Omega_1}} \in H^1(\Omega_2 \setminus \overline{\Omega_1})$. But this is in contradiction to the fact that $G_\gamma(\cdot, y)|_{\partial\Omega_1} \notin L^2(\partial\Omega_1)$ when $n > 2$ by (3.14). When $n = 2$ we reach a contradiction by first extending $G_\gamma(\cdot, y)|_{\Omega_2 \setminus \overline{\Omega_1}}$ as H^1-function to some neighborhood of y and then using Theorem 7.15 in [G-T] together with (3.14).

The case (B) is proved in a similar fashion; one just has to use Kato's theorem and UCP as in the beginning of the proof of Theorem 3.4 to see that the far field pattern uniquely determines the solution of the Schrödinger equation. For the singular behavior of $G^+_{k,q}(\cdot, y)$ at y we refer to [Se] or [Si]. We note that for dimensions $n = 2, 3$ locally H^2-functions are continuous by the Sobolev imbedding theorem and for dimensions $n > 3$ the singularity $G^+_{k,q}(x, y) \approx |x - y|^{2-n}$ doesn't allow it to belong to $L^2_{loc}(\mathbf{R}^n)$.

□

We still have to make sure that the conditions in Theorem 3 are valid in (A) and (B):

Proposition 3.6 *In the problem (A) condition (1.4) is valid for* $\mathcal{S} = \mathcal{S}_1$ *defined by equation (3.11).*

In the problem (B) condition (1.4) is valid for $\mathcal{S} = \mathcal{S}_{k,q}$ *or* $\mathcal{N}^\sigma_{k,q}$ *under the assumption that* k^2 *is neither a Dirichlet nor a Robin eigenvalue of* $-\Delta + q$ *in* Ω.

Proof. In the case (A) the ellipticity of γ implies that there exists $c > 0$ s.t. for every $g \in H^{-\frac{1}{2}}(\partial\Omega_2)$

$$|\langle \mathcal{S}_1 g, g\rangle| = |\langle \mathcal{G}_\gamma \tau_1^* g, \tau_1^* g\rangle| \geq c\|\tau_1^* g\|$$

and because the surjectivity of τ_1 implies the continuity of $(\tau_1^*)^{-1}$, we have the claim.

In the case (B) we use the resolvent equation (3.45), the compactness of q as a mapping from H^1_{comp} into H^{-1}_{comp} and Theorem 2 in [C] to write the operators $\mathcal{S}_{k,q}$ and $\mathcal{N}^\sigma_{k,q}$ as compact perturbations of positive operators. Then Lemma 4.2 in [G-K] combined with the facts that $Im\langle R_0^+ g, g\rangle = 0 \Rightarrow \widehat{g}|_{kS^{n-1}} = 0$ (as in the proof of Theorem 3.4) and that k^2 is not an interior eigenvalue proves the claim.

□

References

[A] Agmon, S.: Spectral properties of Schrödinger operators and scattering theory. Ann. Scuola Norm. Sup. Pisa (4) 2, 151-218 (1975).

[B] Brühl, M.: Explicit characterization of inclusions in electrical impedance tomography, SIAM J. Math. Anal. 32 (6), 1327-1341 (2001).

[C] Costabel, M.: Boundary integral operators on Lipschitz domains: elementary results. SIAM J. Math. Anal. 19 (3), 613-626 (1988).

[C-K] Colton, D. and Kress, R.: Inverse acoustic and electromagnetic scattering theory, Springer-Verlag (1992).

[G-K] Grinberg, N. and Kirsch, A.: The linear sampling method in inverse obstacle scattering for impedance boundary conditions, Journal of Inverse and Ill-Posed Problems 10, 171-185 (2002).

[G-T] Gilbarg, D. and Trudinger, N.S.: Elliptic partial differential equation of second order, Springer-Verlag (1977).

[G-Y] Gâtel, Y. and Yafaev, D.: On solutions of the Schrödinger equation with radiation conditions at infinity: the long-range case, Ann. Inst. Fourier, Grenoble 49 (5), 1581-1602 (1999).

[H] Hörmander, L: The analysis of linear partial differential operators II, Springer-Verlag (1983).

[H2] Hörmander, L: The analysis of linear partial differential operators III, Springer-Verlag (1985).

[I-N] Isakov, V. and Nachman, A.: Global uniqueness for a two-dimensional semilinear elliptic inverse problem, Trans. Amer. Math. Soc. 347 (9), 3375-3390 (1995).

[J-K] Jerison, D. and Kenig, C.: Unique continuation and absence of positive eigenvalues for Schrödinger operators, Ann. of Math. 121, 463-494 (1985).

[K-S] Kaipio, J and Somersalo, E., *Statistical and computational inverse problems.* Applied Mathematical Sciences, 160. Springer-Verlag, New York, 2005. xvi+339 pp.

[K] Kato, T.: Growth properties of solutions of the reduced wave equation with a variable coefficient, Comm. Pure and Appl. Math. 12, 403-425 (1959).

[Ki1] Kirsch, A.: An introduction to the mathematical theory of inverse problems, Springer-Verlag (1996).

[Ki2] Kirsch, A.: Characterization of the scattering obstacle by the spectral data of the far field operator, Inverse Problems 14, 1489-1512 (1998).

[Ki3] Kirsch, A.: New characterizations of solutions in inverse scattering theory, Appl. Anal. 76, 319-350 (2000).

[Ki4] Kirsch, A.: The factorization method for a class of inverse elliptic problems, To appear in Math. Nachrichten.

[K-P] Kirsch, A. and Päivärinta, L.: On recovering obstacles inside inhomogeneities, Math. Meth. Appl. Sci. 21, 619-651 (1998).

[L-P] Lax, P.D. and Phillips, R.S. Scattering Theory, Academic Press, NY (1967).

[Ma] Majda, A: A representation formula for the scattering operator and the inverse problem for arbitrary bodies, Comm. Pure Appl. Math.,30, 165-194 (1977).

[M] McLean, W.: Strongly elliptic systems and boundary integral equations, Cambridge University Press (2000).

[N1] Nachman, A.: Inverse scattering at fixed energy, Proc. 10th International Congress on Math. Phys., Leipzig 1991, edited by K. Schmüdgen, Springer-Verlag (1992).

[N2] Nachman, A.: Global uniqueness for a two-dimensional boundary value problem, Ann. of Math. 143, 71-96 (1996).

[R-S I-IV] Reed, M. and Simon, B.: Methods of modern mathematical physics, Volumes I-IV, Academic Press (1978).

[S] Schechter, M.: Spectra of partial differential operators, North-Holland Publishing Company (1971).

[Se] Serrin, J.: Local behavior of solutions of quasi-linear equations, Acta Mathematica 3, 247-302 (1964).

[Si] Simon, B.: Schrödinger semigroups, Bull. A.M.S. 7 (3), 447-526 (1982).