Analytic Methods for Inverse Scattering Theory

Lassi Päivärinta

University of Oulu, Department of Mathematical Sciences, Linnanmaa,
P.O.Box 3000, FIN-90401 Oulu, Finland, e-mail: lassi@rieska.oulu.fi

Introduction

The purpose of these lectures is to provide basic analytic tools of fixed energy inverse scattering theory. As a model uses we study the inverse scattering problems for time harmonic acoustic and Schrödinger equations. Section 1 describes these two problems. In Section 2 we introduce the Hardy–Littlewood maximal function and define the Sobolev spaces in $\mathbb{R}^n$. At the end of this Section we prove an important characterization of $W^1_p(\mathbb{R}^n)$ due to P. Hajłasz. In the third Section we prove the continuity of $(\triangle + k^2)^{-1}$ for $L^p(\Omega)$ to $L^q(\Omega)$, for $1 \leq p \leq 2 \leq q \leq \infty$ together with an appropriate norm estimate. As a special case $p = q = 2$ we get S. Agmon's result that norm of $(\triangle + k^2)^{-1}$ in this case behaves as $\frac{1}{k}$ for large k.

In Section 4 we deal with Faddeév's Green's functions. Especially we give a new proof of Sylvester's and Uhlmann's norm estimate by using Hajłasz characterization of W^1_p.

Section 5 proves that the scattering amplitude with fixed energy uniquely determines the scattering potential and finally in Section 6 we use the result of previous sections to prove in two dimensions that the Born approximation that linearizes the inverse backscattering problem carries the same singular structure as the original potential.

The prerequisites for these lectures consist of basic knowledge of real analysis, Fourier analysis and distribution theory.

1 Two Scattering Problems

The theory of acoustic wave propagation is developed from the laws of *classical mechanics.* This theory includes the interaction of acoustic waves with matter usually referred as scattering. *Quantum mechanics* describes phenomena of atomic scale. As a non-causal theory it has a completely different character than classical mechanics. Surprisingly, when one writes down the differential equations for the wave function in two-body scattering the resulting equation, called Schrödinger equation, is in striking similarity to the equation for the velocity potential in acoustic scattering from inhomogeneous medium. In fact, if one studies the scattering only with one fixed energy one observes that the equations discussed above are completely equal in mathematical sense. In this section we introduce the basic differential equations for acoustic and Schrödinger scattering and derive the corresponding integral equation usually called as *Lippmann–Schwinger equation.*

Schrödinger Scattering

Let $H_0 = -\triangle$ be the Hamiltonian of energy k^2 in the vacuum. The *two-body Hamiltonian*

$$H = H_0 + q(x) \tag{1}$$

describes the state of the quantum mechanical system through the *Schrödinger equation*

$$(H - k^2)\Psi = 0 \tag{2}$$

The *scattering* of the *incident field* Ψ_i from the potential q is described by

$$\begin{cases} (H - k^2)\Psi = 0, & \text{in } \mathbb{R}^n, \\ \Psi = \Psi_i + \Psi_s, & \text{where } (\triangle + k^2)\Psi_i \equiv 0 \\ \left(\frac{\partial}{\partial r} - ik\right)\Psi_s(x) = o\left(|x|^{\frac{1-n}{2}}\right), & \text{as } |x| = r \to \infty. \end{cases} \tag{3}$$

The last condition for the *scattered field* Ψ_s is called the *Sommerfeld radiation condition* (S.R.C.).

For $q \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $q(x) = O\left(|x|^{-1-\varepsilon}\right)$ the problem (3) is equivalent to the *Lippmann–Schwinger equation*

$$\Psi(x) = \Psi_i(x) - \int_{\mathbb{R}^n} \Phi_+(x-y)q(y)\Psi(y)dy, \tag{4}$$

where Φ_+ is the outgoing fundamental solution of the Helmholtz equation:

$$\Phi_+(x) = \frac{1}{2\pi}\mathcal{F}^{-1}\left(\frac{1}{\xi^2 - k^2 - i0}\right)(x).$$

Here $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Remarks.

1°) Denote by $g(\xi)$ the distribution

$$\langle g, \varphi\rangle = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n} \frac{\varphi(\xi)}{\xi^2 - k^2 - i\varepsilon}d\xi$$

Then it is easy to see that $g \in \mathcal{S}'$ and hence

$$\Phi_+ = \mathcal{F}^{-1}g$$

is well defined.

2°) For $n = 3$

$$\Phi_+ = \frac{e^{ik|x|}}{4\pi\,|x|}$$

3°) One can show that $\Phi_+ \in L^1_{\text{loc}}$ for every $n \geq 2$.

Acoustic Scattering

Let $c(x)$ be the speed of sound that we assume to differ from the background speed of sound c_0 only in the bounded domain $\Omega \subset \mathbb{R}^3$. If $u = u(x,t)$ is the velocity potential then u satisfies the wave equation

$$\triangle u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

If we assume that u is time-harmonic we can write

$$u(x,t) = \mathrm{Re}\left(u(x)e^{i\omega t}\right)$$

and we get for u the equation

$$\triangle u(x) + k^2 n(x) u(x) = 0 \quad \text{in } \mathbb{R}^3, \tag{5}$$

where $k^2 = \frac{\omega^2}{c_0^2}$ and $n(x) := \frac{c_0^2}{c(x)^2}$ is called the *refractive index.*

When there is dissipation in the acoustic medium, the equation (5) still holds but $n(x)$ is complex-valued and $\mathrm{Im}\,(n(x))$ causes the dissipation.

To relate the two above problems we write

$$m = 1 - n$$

and (5) rewrites as

$$(-\triangle - k^2 + q(x))u(x) = 0, \tag{6}$$

where $q(x) = k^2 m(x)$. Thus acoustic equation and Schrödinger equation are formally equivalent. Note however that in acoustic equation the potential q depends on the wave number k and thus yields different asymptotic behavior for the situation as $k \to 0^+$ or $k \to +\infty$.

2 Maximal Functions and Sobolev Spaces

In 1930's C. G. Hardy and J. E. Littlewood introduced the notions of maximal function and proved the fundamental theorem corresponding our Theorem 2.1 below. At the same time S. V. Sobolev developed the theory of distributions (generalized functions) and introduced his famous spaces. It was only in 1990's when P. Hajłasz recognized that the Sobolev spaces $W_p^1(\mathbb{R}^n)$ can be characterized with the help of *Hardy–Littlewood maximal functions.* In this section we prove Hajłasz's characterization theorem.

The starting point is Lebesgue's fundamental theorem:
For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$

$$\lim_{r\to 0^+} \frac{1}{m(B(x,r))}\int_{B(x,r)} f(y)dy = f(x)$$

holds for almost every $x \in \mathbb{R}^n$.

Define

$$M(f)(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\, dy =: \sup_{r>0} \fint_{B(x,r)} |f(y)|\, dy. \tag{7}$$

Here $M(f)$ is called the *Hardy–Littlewood maximal function* of f. It satisfies

$$M(f)(x) \geq |f(x)| \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{8}$$

One can replace the open balls in the definition (7) with open cubes, as well. The fundamental and a little surprising result of Hardy and Littlewood is

Theorem 2.1. *If $f \in L^p(\mathbb{R}^n)$, $1 < p \leq \infty$, then $Mf \in L^p(\mathbb{R}^n)$ and there exists $C > 0$ such that*

$$\|Mf\|_{L^p} \leq C \|f\|_{L^p}$$

and C depends only on p and n.

Proof. C.f. [21] page 5. □

It is easy to show that Lebesgue's theorem follows from Theorem 2.1.

Sobolev Spaces

Definition 2.2. We define for open $\Omega \subset \mathbb{R}^n$ that $f \in W_p^k(\Omega)$, if $D^\alpha f \in L^p$ for every $|\alpha| \leq k$ and then the norm is defined by

$$\|f\|_{W_p^k} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p}$$

We also define Sobolev spaces $H_p^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$ by

$$f \in H_p^s(\mathbb{R}^n) \text{ if and only if } \mathcal{F}^{-1}\left((1+\xi^2)^{s/2} \hat{f}(\xi)\right) \in L^p$$

It is an easy exercise to show that $H_p^k(\mathbb{R}^n) = W_p^k(\mathbb{R}^n)$ for $k = 0, 1, 2, \ldots$

Our main tool for what is coming is the following recent characterization of $W_p^1(\mathbb{R}^n)$ by P. Hajłasz. (c.f. [9])

Theorem 2.3. *For $1 < p \leq \infty$ the function $u \in W_p^1(\mathbb{R}^n)$ if and only if there exists $g \in L^p(\mathbb{R}^n)$ and $C > 0$ such that*

$$|u(x) - u(y)| \leq C\, |x - y|\, (g(x) + g(y))$$

For g we can choose $g = M(|\nabla u|)$.

Before proving Theorem 2.3 we need

Lemma 2.4. *For $f \in L^1_{loc}(\mathbb{R}^n)$ we have*

$$\int_{B(0,\rho)} f(x)\, |x|^{1-n}\, dx \leq C\rho M(f)(0). \tag{9}$$

Proof. First we show that it is enough to show (9) for $\rho = 1$ i.e.

$$\int_{B(0,1)} f(x)\,|x|^{1-n}\,dx \le C\rho M(f)(0) \tag{10}$$

Indeed, if (10) is true define $g(x) = f(x\rho)$. Then

$$\rho^{-n}\int_{B(0,\rho)} f(x)\left|\frac{x}{\rho}\right|^{1-n} dx = \int_{B(0,1)} g(x)\,|x|^{1-n}\,dx \overset{(10)}{\le} CM(g)(0) \tag{11}$$

But

$$M(g)(0) = \sup_{r>0} r^{-n}\int_{B(0,r)} f(x\rho)dx = \sup_{r>0}(r\rho)^{-n}\int_{B(0,r\rho)} f(x)dx = M(f)(0).$$

To show (10) we write with $R_k = \left\{2^{-k} \le |x| \le 2^{1-k}\right\}$

$$\begin{aligned}\int_{B(0,1)} f(x)\,|x|^{1-n}\,dx &\le \sum_{k=1}^{\infty} 2^{n(1-k)}\int_{R_k} f(x)2^{-k(1-n)}dx \\ &\le CM(f)(0)\Big(\sum_{k=1}^{\infty} 2^{-k+n}\Big) \le CM(f)(0).\end{aligned}$$

□

The next lemma shows how the oscillation from the mean value can be estimated by the Riesz–potential of the gradient.

Lemma 2.5. *For $u \in W^1_p(\mathbb{R}^n)$ and an open cube $Q \subset \mathbb{R}^n$ one has for some $C > 0$ that*

$$|u(x) - u_Q| \le C\int_Q \frac{|\nabla u(y)|}{|x-y|^{n-1}}dy$$

where $u_Q = ⨍_Q u(x)dx$.

Proof. For $x, y \in Q$ and $\omega = \frac{y-x}{|x-y|}$ we have

$$u(x) - u(y) = -\int_0^{|x-y|} D_r u(x + r\omega)dr$$

and hence

$$|Q|\,(u(x) - u_Q) = -\int_Q dy\int_0^{|x-y|} D_r u(x + r\omega)dr.$$

Denoting $d = \text{diam } Q$ and writing $V = |D_r u| \chi_Q$ we have

$$\begin{aligned}|u(x) - u_Q| &\leq \frac{1}{|Q|} \int_{|x-y|\leq d} dy \int_0^\infty V(x + r\omega) dr \\ &= \frac{1}{|Q|} \int_0^\infty dr \int_{|\omega|=1} dS(\omega) \int_0^d d\rho V(x + r\omega)\rho^{n-1} \\ &= \frac{d^n}{n\,|Q|} \int_0^\infty \int_{|\omega|=1} V(x + r\omega) dS(\omega) dr \\ &= C \int_Q \frac{V(y)}{|x-y|^{n-1}} dy \leq C' \int_Q \frac{\nabla u(y)}{|x-y|^{n-1}} dy.\end{aligned}$$

□

Proof of Theorem 2.3. To prove the necessity of the condition in Theorem 2.3 assume $u \in W_p^1(\mathbb{R}^n)$. Now by Lemma 2.5

$$\begin{aligned}|u(x) - u(y)| &\leq |u(x) - u_Q| + |u(y) - u_Q| \\ &\leq C \int_Q |\nabla u(w)| \left(\frac{1}{|w-x|^{n-1}} + \frac{1}{|w-y|^{n-1}} \right) dw\end{aligned}$$

for any cube Q such that $x, y \in Q$. It is enough to prove that

$$\int_Q \frac{|\nabla u(w)|}{|w-x|^{n-1}} dw \leq C(\text{diam } Q) M(|\nabla u|)(x).$$

This follows from Lemma 2.4.

To prove that the condition is sufficient assume that such a $g \in L^p$ exists: Now if $e_j = (0, \ldots, 1, \ldots, 0)$ we have

$$\left\| \frac{u(x + he_j) - u(x)}{h} \right\|_{L^p} \leq C \left\| g(x + he_j) + g(x) \right\|_{L^p} \leq C'$$

where C' is independent of h.

Since $1 < p \leq \infty$ the space L^p is a dual of $L^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and by Alaoglu's theorem there exists a sequence $\{h_l\}_{l=1}^\infty$, $h_l \to 0^+$ such that

$$\frac{u(x + h_l e_j) - u(x)}{h_l}$$

converges to a function $f \in L^p$. On the other hand

$$\frac{u(x + he_j) - u(x)}{h}$$

converges to $D_j u$ in weak topology of $\mathcal{D}'$. By uniqueness of the limit $D_j u = f \in L^p$. Thus $u \in W_p^1$. □

One should notice that the Hajłasz characterization 2.3 allows one to define Sobolov spaces $W_p^1(X)$ on arbitrary metric space X. For this theory we refer to a recent book of J. Heinonen [10].

3 Mapping Properties of $(\triangle + k^2)^{-1}$

In [1] Smuel Agmon proved that $(\triangle + k^2)^{-1}$ acts between weighted L^2-spaces as a bounded operator, where the operator norm can be estimated by c/k. In this section we prove a version of Agmon's result as well as its generalization to L^p-spaces by using Maximal function techniques described in the previous section.

The Helmholtz equation

$$(\triangle + k^2)u = 0, \text{ in } \mathbb{R}^n$$

has outgoing and incoming fundamental solutions

$$G_\pm(x) = \mathcal{F}^{-1}\left(\frac{1}{\xi^2 - k^2 \mp i0}\right)(x).$$

They are characterized by

$$(\triangle + k^2)G_\pm = -\delta$$

and the radiation condition

$$\left(\frac{\partial}{\partial r} \mp ik\right) G_\pm(x) = o\left(|x|^{\frac{1-n}{2}}\right), \text{ as } |x| = r \to \infty.$$

We study the linear operators $\mathbf{G}_\pm$ defined by

$$\mathbf{G}_\pm f = G_\pm * f.$$

We define the weighted spaces L^p_δ by

$$L^p_\delta = \{ f \in L^p_{\text{loc}}(\mathbb{R}^n) \mid (1 + |x|^2)^{\delta/2} f \in L^p \}.$$

Next theorem is an extension of Agmon's classical result to weighted L^p-spaces.

Theorem 3.1. *For $1 < p \leq 2 \leq q < \infty$, and $\delta \geq 1$ the operators $\mathbf{G}_\pm$ can be extended as bounded operators from L^p_δ to $L^q_{-\delta}$ and*

$$\|\mathbf{G}_\pm\|_{L^p_\delta \to L^q_{-\delta}} \leq \frac{c}{k^{1-s}}$$

where $s = n(\frac{1}{p} - \frac{1}{q})$.

Proof. We will first prove the case $p = q = 2$ i.e. the Agmon's case. Let $f, g \in C_0^\infty(\mathbb{R}^n)$. Now by Parseval's identity

$$(\mathbf{G}_\pm f, g) = \int_{||\xi|-k|<k/2} \frac{\hat{f}(\xi)\overline{\hat{g}(\xi)}}{\xi^2 - k^2 \mp i0} d\xi + \int_{||\xi|-k|>k/2} \frac{\hat{f}(\xi)\overline{\hat{g}(\xi)}}{\xi^2 - k^2 \mp i0} d\xi$$

We denote the first integral by I_1 and the second integral by I_2. Since

$$\sup_{||\xi|-k|>k/2} \frac{1}{||\xi|-k|\,(|\xi|+k)} \leq \frac{2}{k^2}$$

we get an estimate

$$|I_2| \leq \frac{c}{k^2} \|\hat{f}\|_{L^p} \|\hat{g}\|_{L^{p'}} . \tag{12}$$

We estimate I_1 by making the change of variables

$$\xi^* = \xi + 2(k - |\xi|)\hat{\xi} = 2k\hat{\xi} - \xi.$$

This maps the domain $\{k/2 < |\xi| < k\}$ to the domain $\{k < |\xi| < 3k/2\}$. The Jacobian for this change of variables is

$$J(\xi) = \left(\frac{2k}{|\xi|} - 1\right)^{n-1}$$

and hence

$$I_1 = \int_{k<|\xi|<3k/2} \left(\hat{f}(\xi)\hat{g}(\xi)m_\pm(\xi) + \hat{f}(\xi^*)\hat{g}(\xi^*)J(\xi)m_\pm(\xi^*)\right) d\xi,$$

where $m_\pm = \frac{1}{\xi^2 - k^2 \pm i0}$. The trick here is to use the fact that $m_\pm$ have different signs inside and outside $\{|\xi| = k\}$. Hence it is not difficult to show that for $k < |\xi| < 3k/2$

$$|m_\pm(\xi) + m_\pm(\xi^*)J(\xi)| \leq \frac{c}{k^2} \quad \text{and} \tag{13}$$

$$|m_\pm(\xi) + m_\pm(\xi^*)| \leq \frac{c}{k^2}. \tag{14}$$

Hence

$$|I_1| \leq \frac{c}{k^2} \int_{k<|\xi|<3k/2} |\hat{f}(\xi)\hat{g}(\xi)| d\xi + \int_{k<|\xi|<3k/2} |\hat{f}(\xi)\hat{g}(\xi) - \hat{f}(\xi^*)\hat{g}(\xi^*)| m_\pm(\xi) d\xi. \tag{15}$$

Since $\|f\|_{L^2_\delta} = \|\hat{f}\|_{W_2^\delta}$ we choose $\delta = 1$ which makes it possible to use Hajłasz' Theorem (Theorem 2.3):

$$m_\pm(\xi)|\hat{f}(\xi) - \hat{f}(\xi^*)| \leq \frac{c}{|\xi| + k} \frac{|\xi - \xi^*|}{||\xi| - k|} |\tilde{f}(\xi) + \tilde{f}(\xi^*)| \leq \frac{c}{k}(\tilde{f}(\xi) + \tilde{f}(\xi^*)) \tag{16}$$

where $\tilde{f} = M(|\nabla \hat{f}|)$. A similar estimate for $m_\pm(\xi)\,|\hat{g}(\xi) - \hat{g}(\xi^*)|$ and estimate (14) yield

$$|I_1| \leq \frac{c}{k}\,\|\hat{g}\|_{W_2^1}\,\|\hat{f}\|_{W_2^1} \leq \frac{c}{k}\,\|g\|_{L^2_\delta}\,\|f\|_{L^2_\delta}. \tag{17}$$

By (12) and (17) for all $f, g \in C_0^\infty$

$$|(\mathbf{G}_\pm f, g)| \leq \frac{c}{|k|}\|f\|_{L^2_\delta}\,\|g\|_{L^2_\delta}\,.$$

By density of C_0^∞ in L^2_δ we are through with L^2-case.

To prove the claims for L^p-spaces we make use of Bessel potential operator J^s that is defined by

$$J^s f(x) = \mathcal{F}^{-1}\left((1+|\xi|^2)^{s/2}\hat{f}(\xi)\right)(x)$$

for every $s \in \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ (c.f [3]). Next we choose $s_1 > 0$, $s_2 > 0$ such that $s_1 + s_2 = s$. We estimate as in the L^2-case but instead of (16) we write

$$\begin{aligned}
&\int |(\hat{f}(\xi) - \hat{f}(\xi^*))\hat{g}(\xi)m_\pm(\xi)|d\xi \\
&\qquad \leq \int (1+|\xi|)^s(\widehat{J^{-s_1}f}(\xi) - \widehat{J^{-s_1}f}(\xi^*))\widehat{J^{-s_2}g}(\xi)m_\pm(\xi)d\xi \\
&\qquad + \int \left|((1+|\xi|)^{-s_1} - (1+|\xi^*|)^{-s_1})\hat{f}(\xi^*)\hat{g}(\xi)(1+|\xi|)^{s_2}\right| d\xi
\end{aligned} \tag{18}$$

The second term is readily estimated by using the mean value theorem

$$\frac{c}{k}\int \frac{|\hat{f}(\xi)\hat{g}(\xi)|}{|\xi|+k}d\xi \leq \frac{c}{k}\,\|f\|_{H^{-s_1}}\,\|g\|_{H^{-s_2}}\,, \tag{19}$$

since $s_1 + s_2 = s < 1$. To take care of the first term in (18) we use the Hajłasz trick to obtain that it can be estimated by

$$\begin{aligned}
&\frac{c}{k^{1-s}}\int M(|\nabla \widehat{J^{-s_1}f}|)(\xi)|\widehat{J^{-s_2}g}(\xi)|d\xi \\
&\quad \leq \frac{c}{k^{1-s}}\left\|xJ^{-s_1}f(x)\right\|_{L^2}\left\|J^{-s_2}g\right\|_{L^2}\,,
\end{aligned} \tag{20}$$

By Sobolev embedding

$$\begin{aligned}
&L^p \hookrightarrow H^{-s_1} \quad \text{for} \quad s_1 \geq \frac{n}{p} - 1 \quad \text{and} \\
&L^{q'} \hookrightarrow H^{-s_2} \quad \text{for} \quad s_2 \geq 1 - \frac{n}{q}
\end{aligned} \tag{21}$$

Since

$$\mathcal{F}\left(xJ^{-s_1}f(x)\right)(\xi) = \nabla\left((1+|\xi|^2)^{-s_1/2}\hat{f}(\xi)\right)$$

the claim follows from (19), (20), (21) and similar estimate for

$$\int |g(\xi) - g(\xi^*))| \, |\hat{f}(\xi)| \, |m_\pm(\xi)| \, d\xi.$$

□

The L^2-result was proven in 1975 by S. Agmon in [1]. The L^p-result is not optimal. This can be seen in the case when q is the dual exponent of p by interpolating the result of Kenig, Ruiz and Sogge [14] with Agmon's result. It is an open question what are the optimal L^p-results in weighted spaces or on a bounded domains.

4 Faddeev's Green's Function

In this Section we study mapping properties of exponentially growing Green's functions for Helmholtz equation. These Green's functions were first introduced by Faddeev [8] and later rediscovered by Nachman and Ablowitz [15], Beals and Coifman [2] and Sylvester and Uhlmann [22]. We will use the proved estimates in the next section to prove the uniqueness of the solution for inverse scattering problem.

We start by considering the planewave

$$u(x) = e^{i\xi\cdot x} = e^{ik\theta\cdot x}, \quad k = |\xi| \text{ and } \theta = \frac{\xi}{|\xi|},$$

where $\xi \in \mathbb{R}^n$. We see that u satisfies the Helmholtz equation

$$(\triangle + k^2)u = 0.$$

This remains true if we replace $\xi \in \mathbb{R}^n$ with $\rho \in \mathbb{C}^n$ satisfying $\rho \cdot \rho = k^2$. Especially, if $\rho \cdot \rho = \rho_1^2 + \cdots + \rho_n^2 = 0$, $\rho_i \in \mathbb{C}$, the functions

$$u_\rho(x) = e^{i\rho\cdot x}$$

are harmonic. It was the idea of Calderón to use these functions to study inverse problems and, in particular, to show that the products of harmonic functions are dense in $L^p(\Omega)$.

We go to a different direction by studying Green's functions for Helmholtz equation of the form

$$G(x) = e^{i\rho\cdot x} g(x), \quad \rho \cdot \rho = k^2$$

where g is regular enough, say $g \in \mathcal{S}'(\mathbb{R}^n)$. Now

$$(\triangle + k^2)G = -\delta$$

implies

$$(\triangle + 2i\rho \cdot \nabla)g = -e^{-i\rho\cdot x}\delta = -\delta. \tag{22}$$

Hence by taking the Fourier transform we have

$$g(x) = \mathcal{F}^{-1}\left(\frac{1}{\xi^2 + 2\rho\cdot\xi}\right)(x).$$

In low dimensions g is quite regular and especially for $n = 2$ and $n = 3$ one has that $g \in L^\infty + L^2$.

We are interested in the mapping properties of the convolution operator

$$\mathbf{G}_\rho \;:\; f \mapsto g * f = \mathcal{F}^{-1}\left(\frac{\hat{f}}{\xi^2 + 2\rho\cdot\xi}\right)$$

Theorem 4.1. *The operator $\mathbf{G}_\rho$ can be extended as a bounded operator from L^p_δ to $L^q_{-\delta}$, for $1 < p \le 2 \le q < \infty$ and $\delta \ge 1$ with norm estimate*

$$\|\mathbf{G}_\rho\|_{L^p_\delta \to L^q_{-\delta}} \le \frac{c}{|\rho|^{1-s}} \tag{23}$$

where $s = n(\frac{1}{p} - \frac{1}{q})$.

Proof. We assume for simplicity that $k = 0$ i.e. $\rho\cdot\rho = 0$. The general case follows analogously. Without a loss of generality we can assume that $\rho = (t, 0, \dots, it)$, $t > 0$.

We denote $\xi'' = (\xi_2, \dots, \xi_{n-1})$ and $\xi = (\xi_1, \xi'', \xi_n)$. Now for $f, g \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned}(\mathbf{G}_\rho f, g) &= \int \frac{\overline{\hat{g}}(\xi)\hat{f}(\xi)}{(\xi_1 + t)^2 + |\xi''|^2 - t^2 + 2it\xi_n + \xi_n^2}d\xi \\ &= \int \frac{\overline{\hat{g}}_t(\xi)\hat{f}_t(\xi)}{\xi_1^2 + |\xi''|^2 - t^2 + 2it\xi_n + \xi_n^2}d\xi,\end{aligned} \tag{24}$$

where $g_t(x) = e^{itx_1}g(x)$ and $f_t(x) = e^{itx_1}f(x)$. Note that

$$\|f_t\|_{L^p_\delta} = \|f\|_{L^p_\delta}$$

and similarly for g.

By using the change of variables

$$\xi \mapsto (\xi^*, -\xi_n),$$

where $\xi^* = 2t\xi' - \xi'$ and $\xi' = (\xi_1, \xi'') = (\xi_1, \dots, \xi_{n-1})$ one can easily see for

$$m_t(\xi) = \frac{1}{\xi'^2 - t^2 + 2it\xi_n + \xi_n^2}$$

that

$$|m_t(\xi) - m_t(\xi^*, -\xi_n) J_t(\xi)| \leq \frac{c}{t^2} \tag{25}$$

where J_t is the Jacobian of the change. Now exactly the same argument of using Theorem 2.3 as in the proof of Theorem 3.1 yields the claim. □

The estimate (23) for $\delta > \frac{1}{2}$ and $p = 2$ goes back to Sylvester and Uhlmann [22] and was a key tool for the uniqueness of the inverse boundary value problem. Hähner [11] gave a simple proof by using Fourier series instead of Fourier integral for the corresponding periodic operator. The corresponding result for a more general class of partial differential equations but for bounded domains were proved by Isakov [12]. Also Hähner's and Isakov's proofs apply only for L^2-case.

5 Uniqueness of the Inverse Scattering Problem

We recall that scattering problem (3) is equivalent to the Lippmann–Schwinger equation (4). To see that (4) has a unique solution we need Rellich's uniqueness lemma. For the formulation of the lemma and of the inverse scattering problem we introduce the notion of *scattering amplitude* or the *far field* of the scattering solution.

Every outgoing solution u of the Helmholtz equation in $\mathbb{R}^n \setminus \overline{\Omega}$ has the asymptotic behavior

$$u(x) = \frac{e^{ik|x|}}{4\pi |x|^{(n-1)/2}} u_\infty(\hat{x}) + o\left(\frac{1}{|x|^{(n-1)/2}}\right) \tag{26}$$

as $|x| \to \infty$.

For the physical case $n = 3$ this can most easily be seen from Green's formulas and from

$$\Phi_+(x-y) = \frac{e^{ik|x|}}{4\pi |x|} e^{-ik\hat{x}\cdot y} + O\left(\frac{1}{|x|^2}\right)$$

as $|x| \to \infty$.

The function u_∞ is called the *far field* of u. If $u = u_i + u_s$ is the solution of the scattering problem (3) the far field of $u_s(x, k, \theta)$ is denoted by $A(k, \theta, \hat{x})$ and is called the scattering amplitude of (3).

The two basic tools for the existence and uniqueness of the solution for the Lippmann–Schwinger equation (4) are Rellich's lemma and the unique continuation principle. For the proofs of these basic tools for scattering theory we refer to the book of Colton and Kress [4].

Lemma 5.1 (Rellich). *Any outgoing solution u of the Helmholtz equation in $\mathbb{R}^3 \setminus \overline{\Omega}$ with vanishing far field must vanish identically in $\mathbb{R}^3 \setminus \overline{\Omega}$.*

Lemma 5.2 (Unique continuation principle). *Assume that* ψ *satisfies*

$$(\Delta + k^2 - q)\psi = 0$$

for some domain $G \subset \mathbb{R}^n$, $q \in L^\infty(G)$ *and that* ψ *vanishes on some open subset of* G. *Then* ψ *vanishes identically on* G.

We are now ready to prove the existence and uniqueness of the solution for the Lippmann–Schwinger equation (4):

Theorem 5.3. *The Lippmann–Schwinger equation* (4) *has a unique solution in* $L^2_{-\delta}$, *for* $\delta > \frac{n}{2}$ *and* $q \in L^\infty_{comp}$.

Proof. Assume that $supp q \subset \Omega$ where Ω is an open ball of radius $R > 0$ centered at origin. By Fredholm's alternative we need to show that $\mathbf{G}_+ q : L^2(\Omega) \to L^2(\Omega)$ is compact and that the homogeneous equation

$$\Psi(x) = -\int_{\mathbb{R}^n} \Phi_+(x-y)q(y)\Psi(y)dy \tag{27}$$

has only the trivial solution in $L^2(\Omega)$. Here $\mathbf{G}_+ q$ means the operator that first multiplies with q and operates with $\mathbf{G}_+$ to the product. To prove the first claim we observe from the symbol of $\mathbf{G}_+$, that is

$$\hat{\Phi}_+(\xi) = \frac{1}{\xi^2 - k^2 - i0}$$

that $\mathbf{G}_+ : L^2(\Omega) \to H^2(\Omega)$. Since, by other Rellich's theorem, $H^2(\Omega)$ is compactly embedded in $L^2(\Omega)$, we are done in this part of the proof.

For the second part we assume that ψ is the solution of the homogeneous equation (27). We first note that by the Green's formula

$$\operatorname{Im}\left(\int_{|x|=r} \psi \frac{\partial}{\partial\nu}\overline{\psi} ds\right) = \operatorname{Im}\left(\int_{|x|\le r} (|\nabla\psi|^2 - (k^2 - q)\,|\psi|^2))dx\right) = 0$$

for $r > R$. Thus

$$\int_{|x|=r} (\left|\frac{\partial}{\partial\nu}\psi\right|^2 + k^2\,|\psi|^2)ds = \int_{|x|=r} \left|\frac{\partial}{\partial\nu}\psi - ik\psi\right|^2 ds \to 0$$

as $r \to \infty$. Especially

$$\int_{|x|=r} |\psi|^2\, ds = \int_{S^{n-1}} |\psi_\infty(\hat{x})|^2\, ds + o\,(1) \to 0, \quad \text{as } r \to \infty.$$

Thus $\psi_\infty \equiv 0$ and by Lemma 5.1 ψ vanishes in $\mathbb{R}^n \backslash \overline{\Omega}$. By the unique continuation principle ψ vanishes identically and we are through. □

The inverse scattering problem is to determine q from the knowledge of $A(k,\theta,\theta')$. The main theorem here due to Novikov, Nachman and Ramm reads as

Theorem 5.4. *For $q \in L^\infty_{comp}(\mathbb{R}^n)$, $n \geq 3$ and k fixed the scattering amplitude uniquely determines q.*

For the proof we need two lemmas.

Lemma 5.5. *For $|\rho|$ large there exists solution u of $(\triangle + k^2 - q)u = 0$ of the form*

$$u = e^{i\rho\cdot x}(1 + R_\rho)$$

where

$$R_\rho \in L^2(\Omega) \quad \text{and} \quad \|R_\rho\|_{L^2(\Omega)} \leq \frac{c}{|\rho|}$$

for every bounded domain Ω with supp $q \subset \Omega$.

Lemma 5.6. *If $u_i \in H^2(\overline{\Omega})$ are any solutions of $(\triangle + k^2 - q_i)u_i = 0$ and $A_{q_1} = A_{q_2}$ then*

$$\int (q_1 - q_2)u_1 u_2 dx = 0 \tag{28}$$

Proof of Theorem 5.4. Assume $A_{q_1} = A_{q_2}$ and let Ω be a domain such that *supp* $q_i \subset \Omega$, $i = 1,2$.

Take $\xi \in \mathbb{R}^n$ arbitrary. We will show that $\hat{q}_1(\xi) = \hat{q}_2(\xi)$ which implies the claim by Fourier inversion theorem. By free choice of coordinate system we may assume that ξ is of the form

$$\xi = (a, 0, \ldots, 0)$$

Choose

$$\begin{aligned} \rho_1 &= (\frac{a}{2}, i\sqrt{M^2 + \frac{a^2}{4}}, 0, \ldots, 0, M) \quad \text{and} \\ \rho_2 &= (\frac{a}{2}, -i\sqrt{M^2 + \frac{a^2}{4}}, 0, \ldots, 0, -M) \end{aligned} \tag{29}$$

where $M > 0$.

Let u_i be the solution of Lemma 5.5 for $(\triangle + k^2 - q_i)u = 0$ for $\rho = \rho_i$, $i = 1,2$. By plugging these to (28) and by observing that $\rho_1 + \rho_2 = \xi$ we obtain

$$\hat{q}_2(\xi) - \hat{q}_2(\xi) = -\int_\Omega (q_1 - q_2)(x) e^{ix\cdot\xi}\left[R_{\rho_1} + R_{\rho_2}(x) + R_{\rho_1}(x)R_{\rho_2}(x)\right] dx$$

Since the left-hand side does not depend on M and the right-hand side tends to zero by the Lemma 5.5 we are done. □

Note that the proof brakes down for $n = 2$ since the choice (29) is not possible.

Proof of Lemma 5.5. The function R_ρ satisfies the equation

$$(\triangle + 2i\rho\cdot\nabla - q)R_\rho = q.$$

By taking the Fourier transform we see that this is equivalent to the integral equation

$$R_\rho - \mathbf{G}_\rho q R_\rho = \mathbf{G}_\rho q \tag{30}$$

By Theorem 4.1 with $p = 2$ we see that the operator $K = \mathbf{G}_\rho q$ satisfies the norm estimate

$$\|K\|_{L^2(\Omega)\to L^2(\Omega)} \le \frac{c}{|\rho|}$$

where Ω is any domain with *supp* $q \subset \Omega$. Thus for large $|\rho|$ the equation (30) can be uniquely solved in $L^2(\Omega)$ by

$$R_\rho = \sum_{n=0}^{\infty} K^n \mathbf{G}_\rho q.$$

Moreover

$$\|R_\rho\|_{L^2(\Omega)} \le \frac{c}{|\rho|}.$$

□

Proof of Lemma 5.6. We first show that every solution of the Schrödinger equation can be approximated in $L^2(\Omega)$ by linear combinations of scattering solutions. More exactly we show that for $f \in H^2(\Omega)$

$$(\triangle + k^2 - q)f = 0 \quad \text{in } \Omega$$

implies

$$f \in \overline{\mathrm{sp}}\ \{\ \psi(x,k,\theta) \mid \theta \in S^{n-1}\ \} \tag{31}$$

where $\overline{\mathrm{sp}}\ A$ means the closed linear span of the set A.

The proof of Theorem 5.3 implies the existence of the outgoing Green's function $G(x,y)$ for $\triangle + k^2 - q$ satisfying

$$(\triangle_x + k^2 - q)G(x,y) = -\delta(x-y) \quad \text{and} \tag{32}$$

$$G(x,y) = \psi(y,k,-\hat{x})\frac{e^{ik|x|}}{4\pi\,|x|^{(n-1)/2}} + o\left(\frac{1}{|x|^{(n-1)/2}}\right) \tag{33}$$

Indeed a solution of

$$G(x,y) = \Phi_+(x-y) - \int_{R^n} \Phi_+(x-z)q(z)G(z,y)dz \tag{34}$$

is such a solution. The same argument as in the proof of Theorem 5.3 yields the uniqueness and existence of the solution of (34). Moreover the asymptotics

$$\Phi_+(x-z) = \frac{e^{ik|x|}}{4\pi\,|x|^{(n-1)/2}} e^{-i\hat{x}\cdot z} + o\left(\frac{1}{|x|^{(n-1)/2}}\right)$$

gives (33) immediately after we have shown

$$\int e^{i\theta\cdot z} q(z) G(z,y)dz = \int \Phi_+(x-z)q(z)\psi(z,k,\theta)dz. \tag{35}$$

Now assume that (31) is not true. Then there exists a function $g \neq 0$ and a sequence of $\{g_i\} \subset H^2(\Omega)$ with $\|g_i - g\|_{L^2} \to 0$ and

$$\int_\Omega g(x)\psi(x,k,\theta)dx = 0 \quad \text{for every } \theta \in S^{n-1} \text{ and} \tag{36}$$

$$(\triangle + k^2 - q)g_i = 0, \quad \text{for every } i \in \mathbb{N}. \tag{37}$$

Define now

$$w(x) = \int_\Omega G(x,y)g(y)dy$$

Now w is an outgoing solution of the Helmholtz equation. Moreover (33) implies that $w_\infty = 0$. Rellich's lemma implies now that

$$w\big|_{\partial\Omega} = \tfrac{\partial}{\partial\nu} w\big|_{\partial\Omega} = 0.$$

Now since

$$(\triangle + k^2 - q)w = g$$

we have

$$\int_\Omega g\overline{g}_i dx = \int_\Omega \left((\triangle + k^2 - q)w\right) g_i dx = \int_{\partial\Omega} \tfrac{\partial}{\partial\nu} w\overline{g}_i - w\tfrac{\partial}{\partial\nu}\overline{g}_i ds$$
$$+ \int_\Omega w(\triangle + k^2 - q)\overline{g}_i dx = 0$$

By letting $i \to \infty$ we have

$$\int_\Omega |g(x)|^2 \, dx = 0$$

which is a contradiction.

By (31) we may now assume that

$$u_1 = \psi_{q_1}(x,k,\theta_1) \quad \text{and} \quad u_2 = \psi_{q_2}(x,k,\theta_2).$$

Since $A_{q_1} = A_{q_2}$ we have by Rellich's lemma that in $\mathbb{R}^n \setminus \overline{\Omega}$

$$\psi_{q_1}(x,k,\theta_i) = \psi_{q_2}(x,k,\theta_i), \quad i = 1,2. \tag{38}$$

Now by Green's second theorem

$$\int_{\Omega}(q_1 - q_2)u_1 u_2 dx = \int_{\Omega}(u_2 \triangle u_1 - u_1 \triangle u_2) dx$$
$$= \int_{\partial\Omega} \left(\psi_{q_2}(x,k,\theta_2)\tfrac{\partial}{\partial\nu}\psi_{q_1}(x,k,\theta_1) - \psi_{q_1}(x,k,\theta_1)\tfrac{\partial}{\partial\nu}\psi_{q_2}(x,k,\theta_2)\right) ds$$

By (38) the last integral vanishes and we are done. □

6 Born Approximation

In this last section we briefly describe how to obtain information about the scattering potential already from the linearized inverse problem. It won't be possible anymore to give complete proofs, so we just try to show the idea and point out what are the essential technical difficulties in the proof. The details can be found in the paper [16]. From now on we assume that $n = 2$. Let $\Psi_-(x) = e^{ik\theta\cdot x}$ be the incoming planewave in (4), and $u(x,k,\theta)$ the corresponding solution of the Lippman–Schwinger equation (4). Let $A(k,\theta,\hat{x})$ be the scattering amplitude. In backscattering we assume that we are looking at the scattered wave in the direction opposite to the incoming direction, i.e. we know the *backscattering data* defined as $D_B = \{ A(k,\theta,-\theta) \mid k \in \mathbb{R},\ \theta \in S^1 \}$. It is not known whether this data determines the potential; the best results are due to Eskin and Ralston [5], [6] and they say that a generic potential q is determined by D_B. We approach the question from a different angle.

If one uses the asymptotics of the outgoing Green's function, one sees that

$$A(k,\theta,\hat{x}) = \int e^{-ik\hat{x}\cdot y} q(y) u(y,k,\theta)\, dy. \tag{39}$$

Evaluating this at the backscattering $\hat{x} = -\theta$ and replacing $u(y,k,\theta)$ by the right–hand side in the Lippman–Schwinger equation one gets

$$A(k,\theta,-\theta) = \hat{q}(-2k\theta) + O\left(|q|^2\right), \tag{40}$$

so it is reasonable to expect that the inverse Fourier–transform

$$q_B(x) = \mathcal{F}^{-1}\left(A(|\xi|/2, -\hat{\xi}, \hat{\xi})\right)$$

is a good approximation for q for potentials which are small in some appropriate norm. Even if q is not small, one could still ask if some features like jumps could be located already from q_B. One can prove the following:

Theorem 6.1. *Assume that a short–range potential q belongs to the weighted Sobolev–space $H^{s_0}_\delta$ for δ large enough. Then there is $\epsilon > 0$ so that $q - q_B \in H^{s_0+\epsilon} + BC$, where BC denotes the space of bounded and continuous functions.*

One actually gets estimates for the ϵ and δ above, and using these it is possible to conclude that if q is piecewise smooth, the location and the height function of the jumps are determined by q_B.

The proof is by iterating the formula for the scattering amplitude (39) with the Lippman–Schwinger equation, as when deducing (40). Doing this N–times one gets

$$A(k,\theta,-\theta) = \hat{q}(-2k\theta) + \sum_{j=1}^{N} \hat{q}_j(-2k\theta) + \hat{q}^R_{N+1},$$

so taking Fourier–transforms

$$q_B - q = \sum q_j + q^R_{N+1},$$

where the terms q_j are $(j+1)$–linear in q and the remainder q^R_{N+1} depends nonlinearly on q; the nonlinearity comes from the nonlinear dependence of the scattering solution u on q. We now need to estimate the smoothness of the terms q_j and q^R_{N+1}. It turns out that the estimates for the bilinear term q_1 are the crucial ones.

A straightforward computation gives

$$q_1(x) = \frac{1}{8\pi^4} \int \frac{\hat{q}(\xi)\hat{q}(\eta)}{\xi \cdot \eta - i0} e^{ix\cdot(\xi+\eta)}\, d\xi\, d\eta.$$

Motivated by this we define the bilinear operator T_1 by

$$T_1(f,g)(x) = \text{pv} \int \frac{f(\xi)g(\eta)}{\xi \cdot \eta} e^{ix\cdot(\xi+\eta)}\, d\xi\, d\eta.$$

Note, that we have changed the integrand by transforming the singularity of the denominator so that it becomes symmetric with respect to $\{\xi \cdot \eta = 0\}$. The difference is easy to compute: for a suitable constant c one has

$$q_1 - cT(\hat{q},\hat{q}) = R(\hat{q},\hat{q}),$$

where

$$R(f,g)(x) = \text{const.} \times \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} f(\xi) g(t\hat{\xi}^{\perp})\, |\xi|^{-1}\, e^{ix\cdot(\xi + t\hat{\xi}^{\perp})}\, dt\, d\xi$$

is easy to estimate by the Cauchy–Schwarz inequality. The result is

Proposition 6.2. *If $q \in H^{s_0}_{\delta}$, where $0 < s_0 < 1$ and $\delta > 0$, then $R(\hat{q},\hat{q}) \in BC$.*

So we are left with estimating T_1. To accomplish this we need characterization of W^1_p due to Hajłasz and the following extension of a maximal inequality due to H. Triebel [23]: If f is a tempered distribution with $\hat{f} \in L^p_\delta$, where $1 < p < \infty$ and $\alpha := \delta - n/p' > 0$, then

$$\sup_{|z|\le 1} |f(x-z)| \le C \max\{\|\hat{f}\|^{n/\alpha}_{L^p_\delta} (Mf)(x)^{1-n/\alpha} \,,\, M(x)\} \text{ a.e.}$$

Taking now the Fourier–transform we get

$$\hat{T}_1(f,g)(\eta) = \text{const.} \times \text{pv} \int \frac{f(\xi)g(\eta-\xi)}{\xi\cdot(\eta-\xi)}\, d\xi.$$

Rather than estimate this we show the idea by estimating

$$S(f,g)(\eta) = \text{pv} \int \frac{f(\xi)g(\eta-\xi)}{\xi\cdot\eta}\, d\xi.$$

The main idea how Hajłasz theorem works with Triebel's maximal function theorem is a bit more trasparent in the estimation of S than T_1. The difference is that in T_1 the integrand is singular on the sphere whereas in S the singularity is on the plane $\{\xi\cdot\eta = 0\}$. One can also estimate T_1 directly, and the analysis is then similar to the proof of Theorem 3.1. Let's proceed with S. Now

$$S(f,g)(\eta) = \frac{1}{|\eta|}\left(\int_{|\xi\cdot\hat{\eta}|\ge 1} + \quad \text{pv}\int_{|\xi\cdot\hat{\eta}|\le 1} \frac{f(\xi)g(\eta-\xi)}{\xi\cdot\eta}\, d\xi\right).$$

The integral over $\{|\xi\cdot\hat{\eta}| \ge 1\}$ is easy to estimate by Young's inequality. We concentrate on the integral over $\{|\xi\cdot\hat{\eta}| \le 1\}$. Let ξ^* be the reflection of ξ with respect to the plane $\{\xi\cdot\eta = 0\}$, i.e. $\xi^* = \xi - 2(\xi\cdot\hat{\eta})\hat{\eta}$. Then the integral over $\{\epsilon < |\xi\cdot\hat{\eta}| \le 1\}$ can be estimated from above by

$$C\int_{\epsilon<\xi\cdot\hat{\eta}\le 1} \frac{|f(\xi)-f(\xi^*)|\,|g(\eta-\xi)| + |f(\xi^*)|\,|g(\eta-\xi)-g(\eta-\xi^*)|}{\xi\cdot\eta}\, d\xi.$$

Let's look at the term involving the differences of f above. Using the Hajłasz's inequality we can bound this by

$$C\int_{0<\xi\cdot\eta<1} (M(|\nabla f|)(\xi) + M(|\nabla f|)(\xi^*))\,|g(\eta-\xi^*)|\, d\xi$$

and then using the extension of Triebel's maximal inequality we get an upper bound

$$C\int M(|\nabla f|)(\xi)M(|g|)(\eta-\xi) + M(|\nabla f|)(\xi^*)M(|g|)(\eta-\xi^*)\, d\xi,$$

which finally can estimated by Young's inequality. The term with the the differences of g can be treated similarly.

The higher order terms q_j and the remainder can be treated as follows. Let K be the operator with Schwartz-kernel

$$k(x,y) = -\left|q(x)\right|^{1/2} G_{k/2}(x-y) q_{1/2}(y),$$

where $q_{1/2} = |q|^{1/2} \operatorname{sgn} q$. Then it is not hard to see that

$$q_j(x) = c \int_{\mathbb{R}^2} \int_0^\infty \int_{|\theta|=1} e^{ik\theta\cdot(x-y/2)} q_{1/2}(y) K^j \left(|q|^{1/2} e^{ik\langle\cdot,\theta\rangle/2} \right) k\, dy\, dk\, d\theta.$$

These terms get smoother as j increases because of the Agmon–estimate: every application of $G_{k/2}$ brings more decay to the Fourier–transform $\hat{q_j}$. The remainder can be treated analogously, and one can prove the following:

Proposition 6.3. *If $q \in L^p_\delta$ with $p > 1$ and $\delta > 2/p'$, then q_j, $q_j^R \in H^t$ for all $t < (j+1/2)p' - 1$.*

The required estimates then follow by using Sobolev–embedding.

We end this section with some comments. There are several works where one studies analogous questions. The original works of Prosser where he shows that for q small enough the backscattering uniquely determines the potential rely also on the Born–series approach. The first paper where rigorous estimates were given for the difference of the original q and its Born–approximation is [20]. This treated a one–dimensional reflection problem. The results were later generalized to higher dimensions in several papers by Päivärinta, Serov and Somersalo [17–19]. The corresponding time–dependent problem was studied using microlocal techniques by Greenleaf and Uhlmann [7], and their work was continued by M. S. Joshi [13]. There results have a slightly different flavor compared to ours. To be able to use microlocal analysis, the potential q is assumed have a certain kind of singularity structure, i.e. it is assumed to be a conormal distribution, and one can then show that the backscattering data determines the complete symbolic expansion of q, so one can recover the location and strength of the singularities.

Acknowledgments

I am grateful for Petteri Piiroinen for preparing the latex-file and making many useful comments. I am indebted to thank Petri Ola for preparing Section 6 of the article and finally I want to thank Matti Lassas and Eero Saksman for many useful discussions.

References

1. S. Agmon, *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **2** (1975), 151–218.
2. R. Beals and R.R. Coifman, *Multidimensional inverse scattering and nonlinear partial differential equations*, Proc. Sympos. Pure Math. **43** (1985), 45–70.
3. J. Bergh and J. Löfström, *Interpolation spaces: An introduction*, Springer-Verlag, Berlin, Heidelberg, New York, 1976.
4. D. Colton and R. Kress, *Inverse acoustic and electromagnetic scatterng theory*, Second edition, Applied Mathematical Sciences, Springer-Verlag, Berlin, 1998.
5. G. Eskin and J. Ralston, *The inverse backscattering problem in 3 dimensions*, Comm. Math. Phys. **124** (1989), 169–215.
6. G. Eskin and J. Ralston, *Inverse backscattering problem in two dimensions*, Comm. Math. Phys. **138** (1991), 451–486.
7. A. Greenleaf and G. Uhlmann, *Recovering singularities of a potential from singularities of the scattering data*, Comm. Math. Phys. **157** (1993), 549–572.
8. L.D. Faddeev, *Increasing solutions of the Schrödinger equation*, Dokl. Akad. Nauk SSSR **165** (1965), 514–517 (Soviet Phys. Dokl. **10** (1966), 1033–1035).
9. P. Hajłasz. *Sobolev spaces on an arbitrary metric space*, Potential Anal. **5** (1996), 403–415,
10. J. Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001, x+140 pp.
11. P. Hähner. *A periodic Faddeev-type solution operator*, J. Differential Equations **128** (1996), no. 1, 300–308.
12. V. Isakov, *Completeness of products of solutions and some inverse problems for pde*, J. Differential Equations **92** (1991), 305–317.
13. M.S. Joshi, *Recovering the total singularity of a conormal potential from backscattering data*, Ann. Inst. Fourier (Grenoble) **5** (1998), 1513–1532.
14. C.E. Kenig, A. Ruiz, and C. Sogge, *Sobolev inequalities and unique continuation for second order constant coefficients differential equations*, Duke Math. J. **55** (1987), 329–348.
15. A.I. Nachman and M.J. Ablowitz, *A multidimensional inverse scattering method*, Studies in Appl. Math. **71** (1984), 243–250.
16. P. Ola, L. Päivärinta and V. Serov, *Recovering singularities from backscattering in two dimensions*, Comm. Partial Differential Equations **26** (2001), 253–271.
17. L. Päivärinta and V. Serov, *Recovery of singularities of a multidimensional scattering potential*, SIAM J. Math. Anal. **29** (1998), 697–711.
18. L. Päivärinta, V. Serov, and E. Somersalo, *Reconstruction of singularities of a multidimensional scattering potential*, Adv. in Appl. Math., **15** (1994), 97–113.
19. L. Päivärinta and E. Somersalo, *Inversion of discontinuities for the Schrödinger equation in three dimensions*, SIAM J. Math. Anal. **22** (1991), 480–499.
20. E. Somersalo, *One-dimensional electromagnetic inverse reflection problem: formulation as a Riemann–Hilbert problem and imaging of discontinuities*, SIAM J. Appl. Math. **49** (1989), no. 3, 944–951.
21. E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, 1970.
22. J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. **125** (1987), 153–169.
23. H. Triebel. *Theory of function spaces*; Akademie–Verlag, Leipzig, 1983.