The Interior Transmission Problem*

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1 Introduction

The interior transmission problem is a boundary value problem arising in inverse scattering theory that, to our knowledge, is not covered by any existing theory in partial differential equations. Nevertheless, the problem is easy to state and a better understanding of conditions under which this problem is well posed would almost surely lead to major advances in inverse scattering theory. Of particular importance is the spectral theory associated with this class of boundary value problems of which essentially nothing is known in more than one dimension. Hence, in view of the central role played by such problems in inverse scattering theory, we are writing this paper in the hope of encouraging other mathematicians to investigate the many unresolved problems associated with boundary value problems of this type.

The plan of our paper is as follows. In the next section we show how the interior transmission problem arises in the scattering of time-harmonic acoustic waves by an inhomogeneous medium of compact support. In particular, we show that the far field operator associated with this scattering problem is injective with dense range provided the wave number is not an eigenvalue of the interior transmission problem. We then consider the case of a spherically stratified inhomogeneous medium and show in Section 3 that such transmission eigenvalues exist and, under certain conditions, uniquely determine the speed of sound in the inhomogeneous medium. In higher dimensions the existence of transmission eigenvalues is unknown and in Section 4 we provide some insight as to why establishing such a result is difficult. However it can be shown that, again under certain conditions, if transmission eigenvalues exist they form at most a discrete set and in Section 5 we establish this fact together with a lower bound for the spectrum. We conclude our presentation by showing in Section 6 how the inhomogeneous interior transmission problem can be used to show that the far field pattern of the scattered field uniquely determines the shape of a penetrable, anisotropic medium. Our hope is that the above discussion will not only convince the reader of the central importance of the interior transmission problem to inverse scattering theory but will also encourage attempts to answer the many open questions associated with boundary value problems of this type.

2 The Far Field Operator and Transmission Eigenvalues

Under appropriate assumptions [5], the scattering of a time-harmonic plane wave by a slowly varying inhomogeneous medium can be modeled by the scattering problem

$$\Delta u + k^2 n(x) u = 0 \quad \text{in } R^3 \tag{2.1}$$

$$u = u^i + u^s \tag{2.2}$$

$$\lim_{r\to\infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 \tag{2.3}$$

where $r = |x|$, the inhomogeneous region is contained inside a ball B, the sound speed $c = c(x)$ is equal to a constant c_0 for $x \in R^3 \setminus B$. The time-harmonic term $e^{-i\omega t}$ has been factored out, the wave number k is defined to be $k = \omega / c_0$,

$$n(x) = c_0^2 / c^2(x)$$

and the incident field u^i is defined to be

$$u^i(x) = e^{ikx\cdot d}$$

where $|d| = 1$. The Sommerfeld radiation condition (2.3) satisfied by the scattered field u^s is assumed to hold uniformly in all directions and the relative sound speed $n(x)$ is assumed to be piecewise continuously differentiable with a jump discontinuity across the smooth boundary ∂D of the simply connected region D where $c(x) \neq c_0$. Under these assumptions it can be shown [5] that the scattered field has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{r} u_\infty(\hat{x}, d) + \mathrm{O}\left(\frac{1}{r^2}\right) \tag{2.4}$$

as $r \to \infty$ where $\hat{x} = x/|x|$ and the wave number k is assumed to be fixed. The function u_∞ is called the *far field pattern* and is known to satisfy [5] the reciprocity relation

$$u_\infty(\hat{x}, d) = u_\infty(-d, -\hat{x}). \tag{2.5}$$

Of basic importance in inverse scattering theory is the *far field operator* $F : L^2(\Omega) \to L^2(\Omega)$, where Ω is the unit sphere, defined by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d) ds(d). \tag{2.6}$$

The superposition principle tells us that $(Fg)(\hat{x})$ is the far field of the solution to (2.1-2.2-2.3) with incident wave u^i equal to the *Herglotz wave function* v_g with kernel g

$$v_g(x) := \int_{\Omega} e^{ikx\cdot d} g(d) ds(d)$$

It can be shown [5], since $n(x)$ is real, that the *scattering operator* S defined by

$$S := I + \frac{ik}{2\pi} F$$

is unitary. A direct consequence of the unitarity of S are the identities:

$$\frac{2\pi}{ik}(F - F^*) = FF^* = F^*F \tag{2.7}$$

from which it follows that F is normal and therefore has identical kernel and cokernel.

If an incident pattern belongs to the kernel of the far field operator, then the corresponding incident wave produces no reflected or scattered wave. The inhomogeneity is *invisible* when illuminated by this pattern. Many inverse scattering techniques, in particular, the linear sampling method, can be guaranteed to work reliably only if this kernel is empty. The interior transmission problem, defined in theorem 1 [5] below will serve as our tool for investigating this kernel. We will let ν denote the unit outward normal to ∂D.

Theorem 1. *The far field operator is injective with dense range if and only if there does not exist a solution $v, w \in C^2(D) \cap C^1(\overline{D})$ of the interior transmission problem*

$$\Delta w + k^2 n(x) w = 0 \qquad in \tag{2.8}$$

$$\Delta v + k^2 v = 0 \qquad in \qquad D \tag{2.9}$$

$$w = v, \; \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad on \qquad \partial D \tag{2.10}$$

such that v is a Herglotz wave function with kernel $g \neq 0$.

Proof. Assume that there are no non-trivial solutions to (2.8) - (2.10) and further that $(Fg)(\hat{x}) = 0$ for $\hat{x} \in \Omega$. Then by Rellich's lemma [5] we have that

$$U^s(x) := \int_{\Omega} u^s(x,d)g(d)ds(d)$$

is equal to zero for $x \in R^3 \backslash \overline{D}$. Hence if $v = v_g$ then from (2.1) - (2.3) we see that v satisfies (2.8)-(2.10) for

$$w(x) := \int_{\Omega} u(x,d)g(d)ds(d).$$

Hence, by the hypothesis of the theorem, $v_g \equiv 0$ and thus by the Fourier inversion theorem for distributions $g = 0$. Hence F is injective.

Conversely, it is easily verified that if there exists a nontrivial solution to (2.8) - (2.10) such that v is a Herglotz wave function with kernel $g \neq 0$ than $(Fg)(\hat{x}) = 0$ for $\hat{x} \in \Omega$ (use the uniqueness of the solution to (2.1)-(2.3)). □

The interior transmission problem (2.8) - (2.10) is the main subject matter of this paper. Values of $k > 0$ such that the interior transmission problem has a nontrivial solution are called *transmission eigenvalues*.

The interior transmission problem depends analytically on the parameter k. We will make use of this by applying the analytic Fredholm theory, but our application will be complicated by the fact that the problem is not elliptic at any k. This is what makes the interior transmission problem different from a standard elliptic eigenvalue problem. We will describe this lack of ellipticity further in section 4, and replace it with an elliptic problem in section 5. We end this section with one simple illustration of this fact. If we set $k = 0$ in (2.8) - (2.10). The system becomes

$$\Delta w = 0 \qquad \text{in } D$$
$$\Delta v = 0 \qquad \text{in } D$$
$$w = v \qquad \text{on } \partial D$$
$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad \text{on } \partial D$$

If (2.8) - (2.10) were elliptic at any k, then the $k = 0$ system would be a relatively compact perturbation and hence have at most a finite dimensional null space. However, $\{v = w\text{equal to any harmonic function}\}$ constitute an infintie dimensional subspace in the kernel of this system.

3 Spherically Stratified Media

In the case of a spherically stratified medium, a considerable amount of information is known about the interior transmission problem (2.8) - (2.10) and in particular transmission eigenvalues. We begin by proving the existence of infinitely many transmission eigenvalues with spherically symmetric eigenfunctions [6] (See also Section 8.4 of [5]). To motivate the hypothesis of the following theorem, suppose for the moment that $n(x)$ is constant. If that constant is equal to one, there is no inhomogeneity, no waves are scattered, and the far field mapping is identically zero, so every k is a transmission eigenvalue. In order to exclude this case, we assume that

$$\frac{1}{a}\int_0^a [n(\rho)]^{\frac{1}{2}} d\rho \neq 1 \tag{3.1}$$

The quantity on the left hand side of (3.1) can be interpreted as one over the harmonic average of the wave speed in the medium, which is the effective wavespeed of a wave with large wavenumber k.

Theorem 2. *Suppose that* $n(x) = n(r), \mathrm{Im}\ n = 0$, $\frac{1}{a}\int_0^a [n(\rho)]^{\frac{1}{2}} d\rho \neq 1$ *and* $n(r) \in C^2(\overline{D})$ *where* $D := \{x : |x| < a\}$*. Then there exist an infinite discrete set of transmission eigenvalues with spherically symmetric eigenfunctions.*

Proof. It suffices to restrict our attention to solutions of (2.8) - (2.10) that depend only on $r = |x|$. Then clearly v must be of the form

$$v(x) = a_0 j_0(kr)$$

where j_0 is a spherical Bessel function of order zero and a_0 is a constant. Writing

$$w(x) = b_0 \frac{y(r)}{r}$$

with a constant b_0, straightforward calculations show that if y is a solution of

$$y'' + k^2 n(r) y = 0$$

satisfying the initial conditions

$$y(0) = 0 \quad , \quad y'(0) = 1$$

then w satisfies(2.8) . We now use the Liouville transformation

$$\zeta := \int_0^r [n(\rho)]^{\frac{1}{2}} d\rho \quad , \quad z(\zeta) := [n(r)]^{\frac{1}{4}} y(r)$$

to arrive at the initial-value problem

$$z'' + [k^2 - p(\zeta)]z = 0 \tag{3.2}$$

$$z(0) = 0 \quad , \quad z'(0) = [n(0)]^{-\frac{1}{4}} \tag{3.3}$$

where

$$p(\zeta) := \frac{n''(r)}{4[n(r)]^2} - \frac{5}{16}\frac{[n'(r)]^2}{[n(r)]^3}.$$

Rewriting (3.2) - (3.3) as a Volterra integral equation

$$z(\zeta) = \frac{\sin k\zeta}{k[n(0)]^{\frac{1}{4}}} + \frac{1}{k}\int_0^\zeta \sin k(\eta - \zeta) z(\eta) p(\eta) d\eta$$

and using the method of successive approximations, we see that the solution of (3.2) - (3.3) satisfies

$$z(\zeta) = \frac{\sin k\zeta}{k[n(0)]^{\frac{1}{4}}} + 0\left(\frac{1}{k^2}\right) \quad \text{and} \quad z'(\zeta) = \frac{\cos k\zeta}{[n(0)]^{\frac{1}{4}}} + O\left(\frac{1}{k}\right)$$

and hence

$$y(r) = \frac{1}{k[n(0)n(r)]^{\frac{1}{4}}} \sin\left(k\int_0^r [n(\rho)]^{\frac{1}{2}} d\rho\right) + O\left(\frac{1}{k^2}\right)$$

and

$$y'(r) = \left[\frac{n(r)}{n(0)}\right]^{\frac{1}{4}} \cos\left(k\int_0^r [n(\rho)]^{\frac{1}{2}} d\rho\right) + O\left(\frac{1}{k}\right)$$

uniformly on $[0, a]$.

The boundary condition (2.10) now requires that

$$b_0 \frac{y(a)}{a} - a_0 j_0(ka) = 0$$

$$b_0 \frac{d}{dr}\left(\frac{y(r)}{r}\right)_{r=a} - a_0 k j_0'(ka) = 0.$$

A nontrivial solution of this system exists if and only if

$$d := \det \begin{pmatrix} \frac{y(a)}{a} & -j_0(ka) \\ \frac{d}{dr}\left(\frac{y(r)}{r}\right)_{r=a} & -kj_0'(ka) \end{pmatrix} = 0. \tag{3.4}$$

Since $j_0(kr) = \sin kr/kr$, from the above asymptotics for $y(r)$ we find that

$$d = \frac{1}{a^2 k}\left\{B \sin k\delta a \cos ka - C \cos k\delta a \sin ka)\right\} + O(1/k^2) \tag{3.5}$$

where

$$B = \frac{1}{(n(0)n(a))^{\frac{1}{4}}} \quad C = \left(\frac{n(a)}{n(0)}\right)^{\frac{1}{4}} \tag{3.6}$$

and

$$\delta = \frac{1}{a}\int_0^a [n(\rho)]^{\frac{1}{2}} d\rho. \tag{3.7}$$

If δ is a natural rational number different from one the claim follows easily since the first term in (3.5) is then a periodic function taking positive and negative values. This fact and (3.5) imply that for k sufficiently large there exists an infinite set of values of k such that (3.4) is true. Each such k is a transmission eigenvalue and this completes the proof of the theorem in this case. If $\delta = 1$ and $C = B$ this term is identically zero but this case is excluded by the assumption. If δ is irrational we can still draw the same conclusion. Indeed in that case this term is an almost-periodic function and the claim follows by applying the definition of almost periodic functions [c.f. [13] Section VI. 5]. □

Since the transmission eigenvalues can be determined from a knowledge of the far field pattern [6], of particular interest in inverse scattering theory is whether or not these eigenvalues determine the relative sound speed $n(r)$. To this end we have the following theorem due to McLaughlin and Polyakov [14]. Before stating the theorem we define

$$A := \int_0^a [n(r)]^{\frac{1}{2}} dr$$

and note that the transmission eigenvalues k_j satisfy [14]

$$k_j^2 = \frac{j^2\pi^2}{(A-a)^2} + O(1)$$

provided that $n(a) = 1$ and that the assumptions of Theorem 2 are satisfied.

Theorem 3. *Assume that $n_1(r)$ and $n_2(r)$ satisfy the assumptions of Theorem 2 and that $n_1(a) = 1$ and $n_2(a) = 1$. Define A_i by*

$$A_i := \int_0^a [n_i(r)]^2 dr$$

for $i = 1, 2$. Suppose there exists $M > 0$ such that all transmission eigenvalues for n_1 and for n_2 that are greater than M coincide. Then $A_1 = A_2 = A$. Suppose further that there is a common subsequence of the transmission eigenvalues denoted by $k_j^2, j = 1, 2, \ldots,$ such that 1) there exists a positive integer m_0 such that

$$|k_j^2| < \frac{(m + \frac{1}{2})^2\pi^2}{(A-a)^2}$$

for $j = 1, 2, \ldots, m$ and $m \geq m_0$ and 2) for $j > m_0$ all the k_j^2 are real and satisfy

$$|k_j^2| > \frac{(m_0 + \frac{1}{2})^2\pi^2}{(A-a)^2}.$$

Then if $3A < a$ we have that $n_1(r) = n_2(r)$ for $0 \leq r \leq a$.

An algorithm for using the transmission eigenvalues to determine $n(r)$ can be found in [15]. The algorithm is based on the Gelfand-Levitan integral equation method [9] and its numerical implementation due to Rundell and Sacks [16].

4 Interior Transmission Problem vs. Usual Transmission Problem

In wave propagation the transmission of a wave from an empty space to a medium with different sound speed is described by the transmission problem

$$\Delta w + k^2 n(x) w = 0 \qquad \text{in } D \tag{4.1}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } \mathbb{R}^3 \backslash D \tag{4.2}$$

$$\lim_{r\to\infty} r\left(\frac{\partial v}{\partial r} - ikv\right) = 0 \tag{4.3}$$

$$w = v + \varphi_0 \qquad \text{on } \partial D \tag{4.4}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} + \psi_0 \qquad \text{on } \partial D \tag{4.5}$$

where φ_0 and ψ_0 are prescribed functions. In this section we compare the mathematical structure of (4.1) - (4.5) to that of

$$\Delta w + k^2 n(x) w = 0 \qquad \text{in} \quad D \tag{4.6}$$

$$\Delta v + k^2 v = 0 \qquad \text{in} \quad D \tag{4.7}$$

$$w = v + \varphi_0 \qquad \text{on} \quad \partial D \tag{4.8}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} + \psi_0 \qquad \text{on} \quad \partial D. \tag{4.9}$$

Notice that both systems (4.1) -(4.5) and (4.6) -(4.9) depend analytically on k. We will use the Calderón projectors below to see that the former system is elliptic, varying k induces a relatively compact perturbation and the Fredholm theory will apply. A similar analysis will show that the transmission eigenvalue problem is not elliptic, and we will have to work harder (and make stronger assumptions) to apply the Fredholm theory.

To demonstrate this we employ the Calderón projectors

$$P_n = \frac{1}{2}(I + A_n) \quad , \quad \text{where}$$

$$A_n = \begin{pmatrix} -K_n & V_n \\ D_n & K_n^* \end{pmatrix}.$$

Here V_n, K_n are the single layer and double layer operators corresponding to (4.1) and $D_n = -\frac{\partial}{\partial \nu} K_n$. For the definition of these operators we refer to the article of M. Costabel and E. Stephan [8], where the basic properties of A_n and P_n are given. In particular

$$P_n \;:\; H^{1/2}(\partial D) \oplus H^{-1/2}(\partial D) \to H^{1/2}(\partial D) \oplus H^{-1/2}(\partial D)$$

continuously and projects any element of this space to a Cauchy data of a solution $w \in H^1(D)$ of (4.1).

Lemma: The following statements on $\binom{\varphi}{\psi} \in H^{1/2}(\partial D) \oplus H^{-1/2}(\partial D)$ are equivalent:

(i) $\binom{\varphi}{\psi}$ is Cauchy data for same $u \in H^1(D)$ satisfying (4.1).

(ii) $A_n\binom{\varphi}{\psi} = \binom{\varphi}{\psi}$.

(iii) There exist $\binom{g}{h} \in H^{1/2}(\partial D) \oplus H^{-1/2}(\partial D)$ such that

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = P_n \begin{pmatrix} g \\ h \end{pmatrix}.$$

Proof. The proof given in [8] is for the case $n \equiv 1$, but it works identically well in the inhomogeneous case. This is so because the proof is based only on the jump relations of the layer potentials and these are the same for n constant and for n inhomogeneous. □

Lemma 4.1 immediately implies the projection property $P_n^2 = P_n$. This is clearly equivalent to

$$A_n^2 = I.$$

In particular, A_n is invertible in $H^{1/2}(\partial D) \oplus H^{-1/2}(\partial D)$.

Similarly one also proves that $\frac{1}{2}(I - A_1)$, where $A_1 = A_n$ for $n \equiv 1$, is a projection operator that projects any element of $H^{1/2}(\partial D) \oplus H^{-1/2}(\partial D)$ to a Cauchy data of a solution to the exterior problem (4.2), (4.3). Hence it follows from the Lemma that (4.1) - (4.5) is equivalent to

$$\frac{1}{2}(I - A_n)\begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix} = 0 \tag{4.10}$$

$$\frac{1}{2}(I + A_1)\begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix} = 0 \tag{4.11}$$

$$\begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix} - \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix}. \tag{4.12}$$

Note that all operators depend on the wave number k, i.e. $A_n = A_n(k)$ etc. Thus, for $\varphi_0 = \psi_0 = 0$, $k > 0$ is an eigenvalue for (4.1) - (4.5) if (4.10) - (4.12), has a non-zero solution $\binom{\varphi}{\psi} \in H^{1/2}(\partial D) \oplus H^{-1/2}(\partial D)$. Moreover, it easily follows from (4.10) - (4.12) that $\binom{\varphi}{\psi}$ then also satisfies the homogeneous integral equation

$$\frac{1}{2}(A_n(k) + A_1(k))\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0 \tag{4.13}$$

The operator in the left hand side of (4.13) is a Fredholm operator of index zero and hence the set of eigenvalues for (4.1) - (4.5) is discrete by the analytic Fredholm theory.

To analyze the interior transmission problem in this manner we introduce conjugate Calderón projectors Q_n and Q_1 defined by

$$Q_n = \frac{1}{2}(I - A_n) \text{ and } Q_1 = \frac{1}{2}(I - A_1)$$

By the Lemma applied separately to $n \equiv 1$ and $n \neq 1$, the number $k > 0$ is a transmission eigenvalue, if and only if the exist non-zero $\binom{\varphi}{\psi}$ such that

$$\frac{1}{2}(I - A_n(k))\begin{pmatrix}\varphi \\ \psi\end{pmatrix} = 0 \text{ and } \frac{1}{2}(I - A_1(k))\begin{pmatrix}\varphi \\ \psi\end{pmatrix} = 0$$

which is equivalent to

$$\ker P_n(k) \cap \ker P_1(k) \neq \{0\}.$$

We say that $k > 0$ is a conjugate transmission eigenvalue if

$$\ker Q_n(k) \cap \ker Q_1(k) \neq \{0\}.$$

We have the following.

Theorem 4. *The following two claims are equivalent.*

(i) $k > 0$ *is either a transmission eigenvalue or a conjugate transmission eigenvalue.*

(ii)
$$\ker(A_n(k) - A_1(k)) \neq \{0\}. \tag{4.14}$$

Proof. Assume first that (i) does not hold and that

$$(A_n(k) - A_1(k))\begin{pmatrix}\varphi \\ \psi\end{pmatrix} = 0.$$

We need to show that

$$\varphi = \psi = 0.$$

To this end define

$$\begin{pmatrix} g \\ h \end{pmatrix} = \frac{1}{2}(I + A_1(k))\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \frac{1}{2}(I + A_n(k))\begin{pmatrix} \varphi \\ \psi \end{pmatrix}. \tag{4.15}$$

Clearly

$$\begin{pmatrix} g \\ h \end{pmatrix} \in \ker Q_n(k) \cap \ker Q_1(k)$$

and by the assumption that (i) does not hold we have that $\binom{g}{h} = 0$. Thus (4.15) implies that

$$\begin{pmatrix} u \\ \psi \end{pmatrix} \in \ker P_n(k) \cap \ker P_1(k)$$

and our assumption yields (4.13). To prove the converse assume that (i) holds. Then there exist non-zero $\binom{\varphi}{\psi}$ such that either

$$(I - A_n)\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0 \text{ and } (I - A_1)\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0 \tag{4.16}$$

or

$$(1 + A_n)\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0 \text{ and } (I + A_1)\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0. \tag{4.17}$$

Clearly either one of (4.16) or (4.17) implies

$$A_n\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = A_1\begin{pmatrix} \varphi \\ \psi \end{pmatrix}.$$

□

The crucial difference between the usual transmission problem and the interior transmission problems lies in the fact that the operator $\frac{1}{2}(A_n(k) + A_1(k))$ appearing in (4.13) is Fredholm but the operator $\frac{1}{2}(A_n(k) - A(k))$ appearing in (4.14) is not. Indeed, one can show in the case where supp $(n - 1) \subset D$, that the operator $\frac{1}{2}(A_n(k) - A(k))$ is infinitely smoothing.

5 The Countability of Transmission Eigenvalues

In contrast to the case of a spherically stratified media, very little is known concerning transmission eigenvalues in the general case of a non-stratified media. Of particular importance in inverse scattering theory is to determine whether or not the set of transmission eigenvalues form a discrete set. In particular, if this set is discrete then the far field operator is injective with dense range for almost all values of the wave number. Under certain hypothesis, the fact that the transmission eigenvalues form a discrete set has been established by Colton, Kirsch and Päivärinta [4] and Rynne and Sleeman [17]. Here we shall outline the proof of this result very similar to that given by Rynne and Sleeman [17].

5.1 The Born Approximation

The far field operator is a differentiable function of $n(x) = 1 + m(x)$. If the scattering is weak (i.e. $|m(x)|$ is small compared to 1) , the far field operator is often replaced by an operator that depends linearly on $m(x)$. This operator is called the Born approximation to the far field operator. The linear dependence makes the analysis of the Born approximation easier than that of the full far field operator. The Born approximation has many equivalent definitions. We make our definition by defining a 1-parameter family of functions n via

$$n(x) = 1 + \epsilon m(x)$$

Let $u_\epsilon = u^i + u^s_\epsilon$ denote the corresponding one parameter family of solutions to (2.1-2.3). The Born approximation to the scattered wave is

$$u^s_B := \frac{d}{d\epsilon}\Big|_{\epsilon=0} u^s_\epsilon$$

and the Born approximation to the far field operator is defined analogously to (2.6), that is,

$$(Bg)(\hat{x}) := \int_\Omega (u_B)_\infty(\hat{x}, d) g(d) ds(d). \tag{5.1}$$

Bg is the far field pattern of u_B, which solves

$$\Delta u_B + k^2 u_B = -k^2 m v_g \quad \text{in } R^3 \tag{5.2}$$

$$u_B = v_g + u^s_B \tag{5.3}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u_B^s}{\partial r} - iku_B^s\right) = 0. \tag{5.4}$$

While the far field operator F is normal, It follows from differentiating (2.7) with respect to ϵ and setting $\epsilon = 0$ that B is actually self-adjoint. In the Born approximation, Theorem 1 becomes

Theorem 5. *The Born far field operator is injective with dense range if and only if there does not exist a solution* $v, w \in C^2(D) \cap C^1(\overline{D})$ *of*

$$\begin{aligned} \Delta w + k^2 w = -k^2 m v \qquad & \text{in} \qquad D \\ \Delta v + k^2 v = 0 \qquad & \text{in} \qquad D \\ w = v,\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad & \text{on} \qquad \partial D \end{aligned}$$

such that v *is a Herglotz wave function with kernel* $g \neq 0$.

The proof is exactly the same as that of Theorem 1. Investigating the existence of transmission eigenvalues for the Born approximation, however, is substantially simpler than the corresponding question for the full far field operator. We shall show that:

Theorem 6. *Suppose that* $m(x) > \delta > 0$ *for all* $x \in D$*, then the kernel of* B *is empty for all* k*, i.e. there are no Born transmission eigenvalues*

Proof. Suppose that g belongs to the kernel of B and that $v = v_g$. If $w = u_B$ and $u = w - v$ then u vanishes, together with its normal derivative, on ∂D and satisfies

$$\left(\Delta + k^2\right) u = -k^2 m v \tag{5.5}$$

If we divide both sides of (5.5) by m, multiply by the complex conjugate of $(\Delta + k^2)\, u$, and integrate, we obtain

$$\begin{aligned} \int_D \frac{1}{m} \left|\left(\Delta + k^2\right) u\right|^2 &= -k^2 \int_D v \overline{(\Delta + k^2)\, u} \\ &= -k^2 \int_D \overline{u} \left(\Delta + k^2\right) v - k^2 \int_{\partial D} \overline{u} \frac{\partial v}{\partial \nu} - \overline{\frac{\partial u}{\partial \nu}} v. \end{aligned}$$

But both the terms on the left hand side are zero, the first because v is a Herglotz wave function and the second because u and its normal derivative vanish on ∂D. Therefore we may conclude that u, which has vanishing Cauchy data on ∂D, satisfies

$$\left(\Delta + k^2\right) u = 0 \quad \text{in} \quad D$$

and hence must be identically zero. Thus by (5.5) v must vanish and hence also g. □

5.2 The Full Far Field Operator

We now return to the full far field operator. It is natural to mimic the proof we have just made for the Born approximation. Not surprisingly, the situation here is more complicated. We will, however, prove the following theorem:

Theorem 7. *Suppose that $m(x) > \delta > 0$ for all $x \in D$, then the set of transmission eigenvalues is a (possibly empty) discrete subset of $\{|k| > \sqrt{\frac{\lambda_0(D)}{\sup_D n}}\}$, where $\lambda_0(D)$ denotes the smallest Dirichlet eigenvalue of $-\Delta$ on D.*

Proof. Suppose that k is a transmission eigenvalue. We rewrite equations (2.8-2.10) to look like equations (5.2-5.4),

$$\begin{aligned} \Delta w + k^2 w = -k^2 m w \qquad & \text{in} \\ \Delta v + k^2 v = 0 \qquad & \text{in} \qquad D \\ w = v, \ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad & \text{on} \qquad \partial D. \end{aligned}$$

We note that $u = w - v$ vanishes, together with its normal derivative, on ∂D and satisfies

$$\left(\Delta + k^2\right) u = -k^2 m w. \tag{5.6}$$

As in the case of the Born approximation, we divide both sides of (5.6) by m and multiply by the complex conjugate of $(\Delta + k^2 n(x))\, z$. In this case we do not choose $z(x) = u$ yet, but we insist that $z \in H_0^2(D)$, in particular, that z and its normal derivative vanish on ∂D. Integrating gives

$$\int_D \overline{((\Delta + k^2 n)\, z)} \frac{1}{m} \left((\Delta + k^2)\, u \right) = -k^2 \int_D w \overline{(\Delta + k^2 n)\, z}$$

$$= -k^2 \int_D \overline{z} \left(\Delta + k^2 n \right) w \qquad (5.7)$$

$$-k^2 \int_{\partial D} \overline{z} \frac{\partial w}{\partial \nu} - \overline{\frac{\partial z}{\partial \nu}} w. \qquad (5.8)$$

As before, the terms in (5.7) and (5.8) are zero. This together with the fact that $(\nabla + k^2)u = (\nabla + k^2 n)u - k^2 m u$ allows us to conclude that

$$0 = \int_D \overline{((\Delta + k^2 n)\, z)} \frac{1}{m} \left((\Delta + k^2)\, u \right) \qquad (5.9)$$

$$= \int_D \overline{((\Delta + k^2 n)\, z)} \frac{1}{m} \left((\Delta + k^2 n)\, u \right) + k^2 (\overline{\nabla z} \cdot \nabla u) - k^4 n \overline{z} u. \qquad (5.10)$$

If we set $z = u$ in (5.10), we can conclude that $(\Delta + k^2 n)\, u = 0$, and hence that u is identically zero whenever the sum of the last two terms is nonnegative. Because

$$\inf_{u \in H_0^2(D)} \frac{\int_D |\nabla u|^2}{\int_D |u|^2} \geq \inf_{u \in H_0^1(D)} \frac{\int_D |\nabla u|^2}{\int_D |u|^2} = \lambda_0(D),$$

we have

$$\int_D k^2 (\overline{\nabla u} \cdot \nabla u) - k^4 n \overline{u} u \geq k^2 ||u||^2_{L^2(D)} \left(\lambda_0(D) - k^2 \sup_D n \right)$$

and the term in parentheses is nonnegative whenever $k^2 \leq \frac{\lambda_0(D)}{\sup_D n}$, i.e. such k cannot be transmission eigenvalues.

To see the discreteness, we rewrite (5.9) as

$$0 = \int_D \overline{(\Delta z)} \frac{1}{m} (\Delta u) + k^2 \overline{z} \frac{1}{m} \Delta u + k^2 \overline{\Delta z} \frac{n}{m} u + k^4 \overline{z} \frac{n}{m} u \qquad (5.11)$$

and begin by finding a lower bound for the first term. Note that

$$
\begin{aligned}
||u||_{H^2}^2 &= ||u||_{L^2}^2 + ||\Delta u||_{L^2}^2 \\
&\leq \left(\frac{1}{\lambda_0(D)^2} + 1 \right) ||\Delta u||_{L^2}^2
\end{aligned}
$$

so that

$$
\begin{aligned}
\int_D \overline{(\Delta z)} \frac{1}{m} (\Delta u) &\geq \left(\inf_D \frac{1}{m} \right) ||\Delta u||_{L^2} ||\Delta z||_{L^2} \\
&\geq \left(\inf_D \frac{1}{m} \right) \left(\frac{1}{\lambda_0(D)^2} + 1 \right)^{-1} ||u||_{H^2} ||z||_{H^2}. \quad (5.12)
\end{aligned}
$$

A consequence of (5.12) is that the self-adjoint map

$$
\Delta \frac{1}{m} \Delta : H_0^2(D) \longrightarrow H^{-2}(D)
$$

has a bounded inverse [1] , specifically,

$$
|| \left(\Delta \frac{1}{m} \Delta \right)^{-1} || \leq \left(\sup_D m \right) \left(\frac{1}{\lambda_0(D)^2} + 1 \right).
$$

We next show that the remaining three terms in (5.11) define (self-adjoint) compact operators from $H_0^2(D)$ to $H^{-2}(D)$.

The second term defines the operator $\frac{1}{m}\Delta$. This is a bounded operator

$$
\frac{1}{m} \Delta : H_0^2(D) \longrightarrow L^2(D)
$$

Because the inclusion of $L^2(D)$ into $H^{-2}(D)$ is compact, we may conclude the compactness of $\frac{1}{m}\Delta$ as an operator from $H_0^2(D)$ to $H^{-2}(D)$.

The third term defines $\Delta \frac{n}{m}$. This is easily seen to be a bounded operator

$$
\Delta \frac{n}{m} : L^2(D) \longrightarrow H^{-2}(D)
$$

The compactness of the inclusion of $H_0^2(D)$ into $L^2(D)$ then implies that $\Delta \frac{n}{m}$ is compact as an operator from $H_0^2(D)$ to $H^{-2}(D)$.

[1] $H^{-2}(D)$ is exactly the dual to $H_0^2(D)$, i.e. the space of bounded linear functionals on $H_0^2(D)$, not the Hilbert space dual.

The fourth term defines the multiplication operator $\frac{n}{m}$ which is compact for similar reasons.

We have shown that, if k is a transmission eigenvalue, then the self-adjoint operator

$$\Delta \frac{1}{m} \Delta + k^2 \Delta \frac{n}{m} + k^2 \frac{1}{m} \Delta + k^4 \frac{n}{m}$$

has non-trivial kernel. This operator is the sum of an invertible operator plus a compact operator which depends analytically on k. Since it is invertible at $k = 0$ the analytic Fredholm theorem guarantees that this can happen at most on a discrete set. $\square$

6 A Uniqueness Theorem for Anisotropic Media

Uniqueness theorems play a central role in inverse scattering theory. In particular, the basic uniqueness question associated with the inverse scattering problem associated with (2.1)-(2.3) is whether or not a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ uniquely determines the relative sound speed $n(x)$. This question was answered affirmatively by Ramm, Novikov and Nachman in 1988 and a simplified proof of this result was given by Hähner in 1996 (See Section 10.2 of [5]). In the case of anisotropic media, the scattering problem (2.1)-(2.3) is replaced by

$$\nabla \cdot A \nabla v + k^2 n v = 0 \quad \text{in} \quad D \tag{6.1}$$

$$\Delta u + k^2 u = 0 \quad \text{in} \quad R^3 \backslash \overline{D} \tag{6.2}$$

$$v - u = 0 \quad \text{on} \quad \partial D \tag{6.3}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial D \tag{6.4}$$

$$u = u^i + u^s \tag{6.5}$$

$$\lim_{r \to \infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 \tag{6.6}$$

where $A = A(x)$ is a matrix with continuously differentiable entries in $\overline{D}$ such that ReA and ImA are symmetric and

$$\overline{\xi} \cdot ImA \xi \leq 0$$

$$\overline{\xi} \cdot Re A\xi \geq \gamma|\xi|^2$$

for all $\xi \in \mathbb{C}^3$ and $x \in \overline{D}$ where γ is a positive constant. It is further assumed that D is bounded, simply connected and has a smooth boundary ∂D with unit outward normal ν,

$$\frac{\partial v}{\partial \nu_A} := \nu \cdot A\nabla v,$$

$u^i(x) = e^{ikx\cdot d}, u^s$ is the scattered field and $n \in C(\overline{D})$ with $Im\ n \geq 0$. The far field pattern u_∞ is again defined by the asymptotic relation (2.4) and the inverse scattering problem associated with (6.1)-(6.6) is to determine D from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$. We also note that in general A and n are *not* uniquely determined by u_∞ [10].

In his seminal paper [12], Hähner showed that for the scattering problem (6.1)-(6.6) the support D of the inhomogeneity is uniquely determined by the far field pattern u_∞ of the scattered field u^s. His proof is based on certain properties of solutions $v, w \in H^1(D)$ to the following *inhomogeneous interior transmission problem* associated with the scattering problem (6.1)-(6.6).

$$\nabla \cdot A\nabla v + k^2 nv = 0 \qquad \text{in} \quad D \tag{6.7}$$

$$\nabla w + k^2 w = 0 \qquad \text{in} \quad D \tag{6.8}$$

$$v - w = f \qquad \text{on} \quad \partial D \tag{6.9}$$

$$\frac{\partial v}{\partial \nu_A} - \frac{\partial w}{\partial \nu} = h \qquad \text{on} \quad \partial D \tag{6.10}$$

where $f \in H^{\frac{1}{2}}(\partial D), h \in H^{-\frac{1}{2}}(\partial D)$ and $H^1(D), H^{\frac{1}{2}}(\partial D), H^{-\frac{1}{2}}(\partial D)$ are the usual Sobolev spaces. In particular, in order to establish uniqueness for the inverse scattering problem described above, the following simple lemma is needed [12]. (See also Section 6.3 of [2]).

Lemma: Assume that either $\overline{\xi} \cdot Re\ A\xi \geq \gamma|\xi|^2$ or $\overline{\xi} Re\ A^{-1}\xi \geq \gamma|\xi|^2$ for some $\gamma > 1$. Let $\{v_n, w_n\} \in H^1(D) \times H^1(D)$ be a sequence of solutions to the inhomogeneous interior transmission problem (6.7)-(6.10) with boundary data $f_n \in H^{\frac{1}{2}}(\partial D)$ and $h_n \in H^{-\frac{1}{2}}(\partial D)$ respectively such that $\{v_n\}$ and $\{w_n\}$ are bounded in $H^1(D)$. Then there exists a subsequence $\{w_{n_k}\}$ which converges in $H^1(D)$.

We are now ready to prove our uniqueness theorem.

Theorem 8. *: Let A_1 and A_2 satisfy the assumptions of the Lemma and $u^i(\hat{x}, d)$ be the far field pattern corresponding to (6.1)-(6.6) for $A = A_i, D = D_i$ and $n = n_i, i = 1, 2$. Then if $u^1_\infty(\hat{x}, d) = u^2_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega, D_1 = D_2$.*

Proof. Let G be the unbounded connected component of $R^3\backslash(\overline{D}_1 \cup \overline{D}_2)$ and $\Phi(x, z)$ the fundamental solution to the Helmholtz equation given by (4.2). Choose a disk Ω_R such that $\overline{D}_1 \cup \overline{D}_2 \subset \Omega_R$ and k^2 is not a Dirichlet eigenvalue for Ω_R. Then since $\{e^{ikx\cdot d} : |d| = 1\}$ is complete in $H^{\frac{1}{2}}(\partial\Omega_R)$ (c.f. Lemma 4.4 of [2]), there exists a sequence $\{u^i_n\}$ in span $\{e^{ikx\cdot d} : |d| = 1\}$ such that for $z \in R^3\backslash\overline{\Omega}_R$

$$||u^i_n - \Phi(\cdot, z)||_{H^{\frac{1}{2}}(\partial\Omega_R)} \to 0$$

as $n \to \infty$ and hence u^i_n approximates $\Phi(\cdot, z)$ in $H^1(\Omega_R)$. Then, since $u^1_\infty = u^2_\infty$, the solutions u^s_1 and u^s_2 of (6.1)-(6.6) for $A = A_i, D = D_i, n = n_i, i = 1, 2$, coincide in G and by the above discussion and the continuous dependence of u^s_i on the boundary data we can now conclude that the solutions $u_1(\cdot, z)$ and $u_2(\cdot, z)$ corresponding to the incident field $\Phi(\cdot, z)$ coincide for fixed $z \in R^3\backslash\overline{\Omega}_R$. Since $u,(\cdot, z)$ and $u_2(\cdot, z)$ are real analytic functions of z for $z \in G$ we now see that $u_1(\cdot, z) = u_2(\cdot, z)$ for all $z \in G$.

Now assume that $\overline{D}_1$ is not included in $\overline{D}_2$ and choose a point $z \in \partial D_1$ and $\epsilon > 0$ such that the points $z_n := z + \frac{\epsilon}{n}\nu(z)$ lie in G for all $n \in N$ where $\nu(z)$ is the unit outward normal to ∂D_1 at z. It is easily seen that $\|\Phi(\cdot, z_n)\|_{H^1(D_1)} \to \infty$ as $n \to \infty$. We now define

$$w^n(x) := \frac{1}{\|\Phi(\cdot, z_n)\|_{H^1(D_1)}}\Phi(x, z_n) \tag{6.11}$$

for $x \in \overline{D}_1 \cup \overline{D}_2$ and let v^n_1, u^n_1 and v^n_2, u^n_2 be the solutions of the scattering problems (6.1)-(6.6) corresponding to A_1, D_1, n_1 and A_2, D_2, n_2 respectively with u^i replaced by w^n. A rather tedious, but straightforward argument [12] now shows that there exists a subsequence $\{u^{n_k}_1\}$ such that

$$\{u^{n_k}_1\} \quad \text{and} \quad \left\{\frac{\partial u^{n_k}}{\partial\nu}\right\}$$

converge in $H^{\frac{1}{2}}(\partial D_1)$ and $H^{-\frac{1}{2}}(\partial D_1)$ respectively.

The proof of the theorem is now easy. Since the functions $v^{n_k}_1$ (corresponding to $u^{n_k}_1$) and w^{n_k} (defined by (6.11)) are solutions of the inhomogeneous interior transmission problem (6.7) - (6.10) for the domain D_1 with boundary data $f = u^{n_k}_1$ and $h = \frac{\partial u^{n_k}_1}{\partial\nu}$, and since the $H^1(D_1)$ norms of $v^{n_k}_1$ and w^{n_k} remain uniformly bounded (by (6.11) and the well-posedness of (6.1) - (6.6), by the Lemma we can select a subsequence of $\{w^{n_k}\}$, again denoted by $\{w^{n_k}\}$, which converges in $H^1(D_1)$ to a function $w \in H^1(D_1)$. As a limit of weak solutions to the Helmholtz equation, $w \in H^1(D_1)$ is a weak solution to the Helmholtz equation. Outside a neighborhood of $z \in \partial D_1, w^{n_k}$ converges uniformly to zero by (6.11) and hence w must be zero in all of D_1 by unique continuation of solutions to the Helmholtz equation. But this contradicts the fact that $\|w^{n_k}\|_{H^1(D_1)} = 1$. Hence $\overline{D}_1$ must be included in $\overline{D}_2$. A similar argument shows that $\overline{D}_2$ must be included in $\overline{D}_1$ and hence $D_1 = D_2$. $\square$

The extension of the above ideas to the inverse scattering problem for Maxwell's equations in an inhomogeneous anisotropic media has been done by Cakoni and Colton [1]. The problem of the countability of the set of transmission eigenvalues for the interior transmission problem for anisotropic media has been investigated by Colton and Päivärinta [7] and Cakoni, Colton and Haddar [3].

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