Clearly

$$\delta(z,\Omega^c) = \sup\{r\in\mathbb{R} \mid z+aw\in\Omega, \text{ if } \delta(w)\le 1, |a|<r\}$$

Above a is scalar. Thus

$$\delta(z,\Omega^c) = \inf_{|w|\le 1} \delta_w(z,\Omega^c)$$

where $\delta_w(z,\Omega^c) = \sup\{r\ge 0 \mid z+\alpha w\in\Omega \text{ if } |\alpha|<r\}$

Thus it is enough to prove the claim for δ_w.

WLG we assume $w=(1,0,\dots,0)$.

To this end let

$$D_k = \{z \mid |z_1|<1 \text{ and } |z_j|<1/k, \ j=2,\dots,n\}$$

$\Rightarrow$ $$\Delta_\Omega^{D_k}(z)\nearrow \delta_w(z,\Omega^c) \quad \text{as } k\to\infty.$$

Let $\varepsilon>0$. It follows (Dini's theorem) that

$$|f(z)|\le \delta_w(z,\Omega^c) \Rightarrow$$
$$|f(z)|\le (1+\varepsilon)\,\Delta_\Omega^{D_k}(z)$$

for large k.

Hence by the first part of the proof

$$|f(\zeta)| \le (1+\varepsilon)\, \Delta_{\Omega}^{D_k}(\zeta) \le (1+\varepsilon)\, \delta_w(\zeta, \Omega^c)$$

for $\zeta \in \hat{K}_\Omega$.

$\boxed{\text{∎}}$

We can now characterize the domains of holomorphy

<u>2.5.5 Theorem</u> If $\Omega \subset \mathbb{C}^n$, then TFAE

(i) Ω is a domain of holomorphy

(ii) $K \subset\subset \Omega \Rightarrow \hat{K}_\Omega \subset\subset \Omega$ and

$$\sup_{z \in K} \frac{f(z)}{\delta(z, \Omega^c)} = \sup_{z \in \hat{K}_\Omega} \frac{f(z)}{\delta(z, \Omega^c)}$$

(iii) $K \subset\subset \Omega \Rightarrow \hat{K}_\Omega \subset\subset \Omega$

(iv) $\exists f \in A(\Omega)$ that cannot be continued beyond Ω

<u>Remark</u> By (iv) we mean that the following is impossible: $\exists\, \Omega_1$ and Ω_2 open s.t.

a) $\emptyset \neq \Omega_1 \subset \Omega_2 \cap \Omega$

b) $\Omega_2 \cap \Omega_1^c \neq \phi$

and $\exists f_2 \in A(\Omega_2)$ s.t.

$$f_2\big|_{\Omega_1} = f\big|_{\Omega_1} .$$

<u>Proof</u> (i) ⟹ (ii) follows from Theorem 2.5.4.

(ii) ⟹ (iii) is trivial. So is (iv) ⟹ (i).

We are left to show (iii) ⟹ (iv):

Let D be 0-cent. ~~polydisc~~ open and $\zeta \in \Omega$.

Let

$$D_\zeta = \zeta + r D$$

where r is as big as possible with $D_\zeta \subset \Omega$.

Let

$M \subset \Omega$ be countable and dense.

It is enough to constr. $f \in A(\Omega)$ s.t. $\forall \zeta \in$ f cannot cont. end. to a nbhd of $\overline{D_\zeta}$.

For this let

$$(\zeta_n)_{n=1}^{\infty}$$

be s.t. $\forall \zeta \in M$, $\#\{n \in \mathbb{N} \mid \zeta_n = \zeta\} = \infty$.

Moreover let $\{K_n\}_{n=1}^{\infty}$ be a seq. of comp. sets s.t.

$K_j \subset K_{j+1} \subset \Omega \quad \forall j$ and

$\bigcup_j K_j = \Omega$.

By assumption (ii) $\hat{K_j} \subset\subset \Omega \quad \forall j$

$\Rightarrow \exists \rho_j$ and $z_j \in D_{\rho_j}$ s.t $z_j \notin \hat{K_j}$

$\Rightarrow \exists f_j \in A(\Omega)$ s.t

$$\begin{cases} f_j(z_j) = 1 \quad \text{and} \\ \sup_{K_j} |f_j| < 1 \end{cases}$$

Replacing f_j with f_j^m we may assum

$$(2.5.4) \qquad \begin{cases} f_j(z_j) = 1 \quad \text{and} \\ \sup_{K_j} |f_j| < 2^{-j} \end{cases}$$

Finally, define

$$(*) \qquad f = \prod_1^\infty (1-f_j)^j$$

If $z \in K_\ell$,

$$\log f(z) = \sum_1^{\ell} j \log(1-f_j) + \sum_{\ell+1}^{\infty} j \log(1-f_j) .$$

and

$$\sum_{\ell+1}^{\infty} j |\log(1-f_j)| \le \sum_{\ell+1}^{\infty} j 2^{-j} < \infty$$

Then (*) conv. unif. on K_ℓ $\Rightarrow$ $f \in A(\Omega)$

Moreover

$$\partial^\alpha f(z_j) = 0 \quad \text{if } |\alpha| < j$$

Since

$$f(z) = (1-f_j(z))^j \prod_{k \neq j} (1-f_k)^k$$

If now $p \in M$ and $N \in \mathbb{N}$, there exist $N_0 > N$ s.t. $\zeta = \zeta_{N_0}$ $\Rightarrow$ $z_j \in D_\zeta$

with

$$\partial^\alpha f(z_j) = 0 \quad \forall\ |\alpha| \leq N$$

Since $\{z_j\}$ has and accum. point z at ∂D

where

$$\partial^\alpha f(z) = 0 \quad \forall\ \alpha$$

$\Rightarrow$ f cannot be analytically cont. over ∂D_ζ.

$\square$

<u>2.5.6 Corollary</u> If Ω is convex then Ω is a domain of holomorphy.

<u>Proof</u> If Ω is convex and $K \subset\subset \Omega$ then $\operatorname{ch} K \subset\subset \Omega$. Hence

$$\widehat{K}_\Omega \subset \operatorname{ch} K \subset\subset \Omega.$$

$\square$

2.5.7 Corollary If Ω_α is domain of holomorphy $\forall \alpha \in A \Rightarrow$

$$\Omega := \operatorname{int} \bigcap_A \Omega_\alpha$$

is a domain of holomorphy

Proof We have for $K \subset\subset \Omega$ that

$$\hat{K}_\Omega \subset \hat{K}_{\Omega_\alpha} \quad \Rightarrow \quad \forall \alpha \in A,$$

$$\delta(\hat{K}_\Omega, \Omega_\alpha^c) = \delta(K, \Omega_\alpha^c) \geq \delta(K, \Omega)$$

$$\Rightarrow \qquad \delta(\hat{K}_\Omega, \Omega^c) \geq \delta(K, \Omega) > 0$$

Hence (iii) in Theorem 2.5.5 holds.

$\square$

2.5.8 Corollary Let Ω be connected and Reinhardt with $0 \in \Omega$. Then FAE

i) Ω is domain of conv. for some $\sum a_\alpha z^\alpha$

ii) Ω is domain of holomorphy

iii)

$\Omega^* = \{\xi \in \mathbb{R}^n \mid (e^{\xi_1}, \dots, e^{\xi_n}) \in \Omega\}$ is open and convex in $\mathbb{R}^n$. Further

$z \in \Omega \Leftrightarrow |z_j| \leq e^{\xi_j} \quad \forall j = 1, \dots, n$

Proof (i) ⇒ (iii) in Theorem 2.4.3.

(ii) ⇒ (i) : Let $f \in A(\Omega)$ as in (iv) in Theorem 2

By Theorem 2.4.5

$$(*) \qquad f(z) = \sum a_\alpha z^\alpha$$

with normal conv. in Ω. Since f cannot be anal. cont. beyond Ω we have that Ω is a domain of conv. for (*).

(iii) ⇒ (ii) : Let $K \subset\subset \Omega$ and k be a finite set s.t.

$$\begin{cases} K \subset \bigcup_{\zeta \in k} D_\zeta \\ \\ \zeta \in k \Rightarrow \zeta_j \neq 0 \end{cases}$$

Here D_ζ is the polydisc

$$\{ z \in \mathbb{C}^n \mid |z_j| \leq |\zeta_j| \}.$$

Assume next $z \in \widehat{K}_\Omega$ with

$$(1) \quad \begin{cases} z_1 \neq 0, \dots, z_j \neq 0 \\ z_{j+1} = \dots = z_n = 0 \end{cases} \qquad \Longrightarrow$$

$$|z_1^{\alpha_1} \dots z_j^{\alpha_j}| \leq \sup_{\zeta \in k} |\zeta_1^{\alpha_1} \dots \zeta_j^{\alpha_j}|$$

Define
$$\lambda_i = \frac{\alpha_i}{\alpha_1 + \dots + \alpha_j}$$
to get

(2) $$\sum_1^j \lambda_i \log|z_j| \le \sup_{\zeta \in K} \sum_1^j \lambda_i \log|\zeta_j|$$

Since $\lambda_i \in \mathbb{Q}$ are arbitrary with $\sum_{j=1}^n \lambda_j = 1$

$\Rightarrow$ we have (2) holds for all

(3) $$\left\{ \begin{array}{l} \lambda_i \in \mathbb{R}_+ \\ \sum_{i=1}^j \lambda_i = 1 \end{array} \right.$$

Set
$$A = \{ y \in \mathbb{R}^j \mid y_i \le \log|\zeta_i| \text{ , for some } \zeta \in K \}$$

Now (2) and (3) imply
$$(\log|z_1|, \dots, \log|z_n|) \in \operatorname{ch} A$$
Since Ω^* is convex we get that
$$|z_i| \le e^{y_i} \text{ , } i = 1, \dots, j \quad \text{for some } y \in \Omega^*$$
$\Rightarrow$
$$|z_i| \le e^{y_i} \quad \forall\, i = 1, \dots, n \quad - \text{"} -$$
$\Rightarrow$ $$\overline{K}_\Omega \subset \Omega$$

$\square$

Thus $\hat{K}_\Omega \subset\subset \Omega$ and Ω is a domain of holomorphy by Theorem 2.5.

$\square$

Corollary: if Ω is a connected Reinhardt domain, then $\forall f \in A(\Omega)$ can be extended to $\Omega_1 \supset \Omega$ which is a domain of holomorphy, which is Reinhardt.

Proof By Th 2.4.5

(*) $\qquad f(z) = \sum a_\alpha z^\alpha$.

Let Ω_1 be a domain of conv. for (*).

Cor 2.5.8 $\Rightarrow$ Ω_1 is a domain of holomorphy and Reinhardt.

$\square$

2.5.9 Def $\Omega \subset \mathbb{R}^n$ is called a tube, if $\exists\ \omega \subset \mathbb{R}^n$ s.t.

$$\Omega = \{ z \mid \mathrm{Re}\, z \in \omega \}$$

The set ω is called the base of Ω

2.5.10 Theorem If $\Omega \subset \mathbb{C}^n$ is connect. tube and $f \in A(\Omega)$ can be extended to $A(\operatorname{ch}\Omega)$

Remark Note that

Ω tube with base ω $\Rightarrow$

$\operatorname{ch}\Omega$ — " — base $\operatorname{ch}\omega$

We need

2.5.11 Lemma Let

$$k = \{(x_1, 0, \dots, 0) \mid 0 \le x_1 \le 1\} \cup \{(0, x_2, \dots, 0) \mid 0 \le x_2$$

and $\Omega \subset \mathbb{C}^n$ a tube with $\omega = b(\Omega)$

$$\omega \subset \operatorname{ch}(k).$$

Set for $0 < \varepsilon < 1/2$

$$K_\varepsilon = \{(x_1, x_2, 0, \dots, 0) \mid x_i \ge 0, \ x_1 + x_2 \le 1, \ x_1 + x_2 - \varepsilon(x_1^2 + x_2^2) \le 1 - \varepsilon\}$$

and

$$k_\varepsilon = \{x + iy \mid x \in k, \ y_1^2 + y_2^2 \le 1/\varepsilon, \ y_3 = \dots = y_n$$

Then $\forall\, \eta \in \mathbb{R}^n$, $A(\Omega)$-hull of $k_\varepsilon + i\eta$ contains $K_\varepsilon + i\eta$.

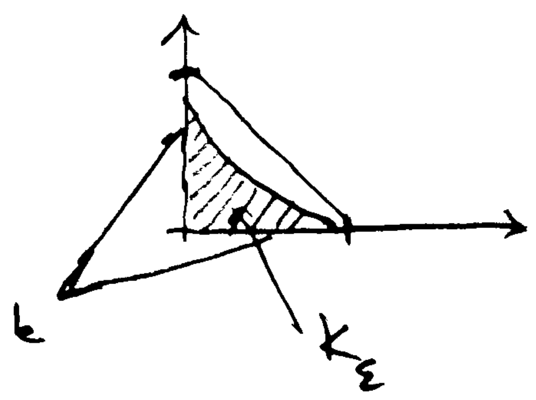
Proof Since Ω is a tube, $f \in A(\Omega) \Rightarrow$

$z \mapsto f(z+i\eta)$ belongs to $A(\Omega)$.

Hence we may assume $\eta = 0$.

We assume $n = 2$.

Now



$$k_\varepsilon = k + iB_{1/\varepsilon}$$

Define
$$M_\varepsilon = \{ z \in \mathbb{C}^2 \mid \operatorname{Re} z_i \geq 0 \quad \operatorname{Re}(z_1 + z_2) \leq 1 \},$$
$$z_1 + z_2 - \varepsilon(z_1^2 + z_2^2) = 1 - \varepsilon \}$$

With $z_j = x_j + iy_j$ we have in M_ε

$$(2.5.5) \quad \begin{cases} x_i \geq 0, \quad x_1 + x_2 \leq 1 \quad \text{and} \\ x_1 + x_2 - \varepsilon(x_1^2 + x_2^2) + \varepsilon(y_1^2 + y_2^2) = 1 - \varepsilon \end{cases}.$$

$$\Rightarrow \quad \begin{cases} x_1^2 + x_2^2 \leq 1 \\ y_1^2 + y_2^2 = \dfrac{1-\varepsilon}{\varepsilon} + (x_1^2 + x_2^2) - \dfrac{x_1 + x_2}{\varepsilon} \leq 1/\varepsilon \end{cases}$$

$\Rightarrow$ M is compact.

We have

$$\frac{\partial}{\partial z_1}\left(z_1+z_2-\varepsilon(z_1^2+z_2^2)\right) = 1-2\varepsilon z_1 \neq 0 \text{ on } M_\varepsilon$$

$\Rightarrow$ $z_2 = h_2(z_1)$ locally, with h_2 analytic. Similarly $z_1 = h_1(z_2)$.

Since $x_1+x_2<1$ on $M_\varepsilon \setminus \{(1,0),(0,1)\}$ we have

$$\partial M_\varepsilon \subset k_\varepsilon \quad \text{i.e. on } \partial M_\varepsilon$$

either $x_1=0$ or $x_2=0$.

Hence for $f\in A(\Omega)$

$$\max_{z\in M_\varepsilon} f(z) = \max_{z\in k_\varepsilon} f(z).$$

Thus $A(\Omega)$-hull of k_ε contains M_ε.

$\Rightarrow$ for $0\le\lambda\le1$ the $A(\Omega)$-hull of λk_ε contains λM_ε. Since $\lambda k_\varepsilon \subset k_\varepsilon$ we have

$$\hat{k}_{\varepsilon,\Omega} \supset \bigcup_{0\le\lambda\le1} \lambda M_\varepsilon = K_\varepsilon$$

$\square$

Proof of Theorem 2.5.10

a) We assume ω is starshaped $\Leftrightarrow$

$$x \in \omega \Rightarrow tx \in \omega, \quad \forall\, 0 \le t \le 1.$$

Then Ω is connected.

$\omega = \omega_1 \cap \omega_2$ is $*$-shaped if ω_i are $*$-shaped, $i=1,2$

Let $\tilde{\omega}$ and $\tilde{\Omega}$ be largest domains s.t.

$\tilde{\omega}$ is $*$-shaped and

$$u \in A(\Omega) \Rightarrow \exists\, U \in A(\tilde{\Omega}) \text{ with } U|_{\Omega} = u.$$

(Take the union of all possible).

We need to show that $\tilde{\omega}$ is convex.

Let $\xi_1, \xi_2 \in \tilde{\omega}$ be lin. indep.

We may assume $\xi_1 = e_1$ and $\xi_2 = e_2$.

Let k be as in Lemma 2.5.11. Fix a 0-centered polydisc D and let $\delta > 0$ be s.t.

$$\Delta^D_{\tilde{\Omega}}(z) > \delta \quad \text{for } z \in k.$$

This is possible since $\tilde{\omega}$ $*$-shaped $\Rightarrow$ $k \subset \tilde{\omega} \subset \Omega$ and k is compact.

Let

$$E = \{ a \in [0,1] \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq a \implies (x_1, x_2, 0, \ldots, 0) \in \tilde{\omega} \}.$$

Clearly

$$0 \in E \quad \text{and}$$

$E \subset [0,1]$ is open since $\tilde{\omega}$ is open, and the distance from

$$k_a = \{ (x_1, x_2) \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq a \}$$

to $\tilde{\omega}^c$ is positive.

Let $a \in E$ and $0 < \varepsilon < 1/2$. Lemma 2.5.11 $\implies$ For any compact $K \subset \mathbb{R}^n$, $A(\tilde{\Omega})$-hull of

$$k + iK \subset \tilde{\Omega}$$

contains

$$L_{a,\varepsilon} = \{ (x_1, x_2, 0, \ldots, 0) \mid x_i \geq 0,\ x_1 + x_2 \leq (1-\varepsilon) a \}.$$

Let $f \in A(\tilde{\Omega})$ and $\zeta \in \mathbb{C}^n$ be such that $\operatorname{Re} \zeta \in L_{a,\varepsilon}$. Then the power series of f converges in

$$\zeta + \delta D \qquad \text{by Lemma 2.5.3}.$$

*) Thus E is closed. Namely if $a_i \to a$, $a_i \in E$,
$L_{a,\varepsilon} \subset L_{a_i, \varepsilon/2}$ for large i.

Hence E is closed and consequently $E = [0,1]$. Thus $\tilde{\omega}$ is convex.

b) Let ω now be open and connected, and $0 \in \omega$. Let $\tilde{\omega}$ be largest s.t.

$\tilde{\omega}$ is $*$-shaped and

$f \in A(\Omega) \Rightarrow \exists\, \tilde{f} \in A(\tilde{\Omega})$ with $\tilde{f} = f$ near 0.

According to a) $\tilde{\Omega}$ is convex.

Assume $\Omega \not\subset \tilde{\Omega}$ $\Rightarrow \exists$ $x_0 \in \omega$ s.t. $x_0 \notin \tilde{\omega}$

Since ω is connected $\Rightarrow \exists$ polygon that connects 0 and x_0. Let x_1 be the first intersection with $\partial\tilde{\omega}$. Then

0 is connected to x_1 with

polygon $\subset \omega \cap \tilde{\omega} \cup \{x_1\}$.

Let ω_1 be a convex nbhd of x_1 s.t.

$\omega_1 \subset \omega$.

Let

$$f_1 = \begin{cases} \tilde{f} & \text{in } \tilde{\omega} + i\mathbb{R}^n \\ f & \text{in } \omega_1 + i\mathbb{R}^n \end{cases}$$

Then f_1 is well-defined: