Let $z \in \mathbb{C}$. If f and g are analytic in some nbhd of z define

$$f \sim g \quad \text{if} \quad f = g \text{ in some nbhd of } z.$$

Denote by f_z the corresponding equiv. class and A_z the space of such f_z's.

<u>Lemma</u> A_z is a ring without 0-divisors

<u>Proof</u> The product and sum are defined

$$f_z + g_z = (f+g)_z$$
$$f_z g_z = (fg)_z$$

Clearly the axioms of ring are satisfied.

Now, if
$$f_z g_z = 0_z$$
then
$$(fg)_z = f_z g_z = 0_z \quad \text{ie}$$

fg is 0 in the neighborhood of z.

Assume that f is 0 in some nbhd of z does not hold $\Rightarrow \exists z_j \to z$ s.t. $f(z_j) \neq 0$

Thus $g(z_j) = 0 \quad \forall j \Rightarrow g(z) = 0$

By Cor. 1.2.10 if $g \not\equiv 0$ then

$$g(w) = w^n v(w) \quad , \quad v(z) \neq 0$$

Now $g(z_j) = 0 \Rightarrow v(z_j) = 0$

This is in contradiction to $v(z) \neq 0$.

Thus

$$\eta_z = 0.$$

$\square$

Let M_z be a quotient field of A_z.

1.4.1. Definition

$$\varphi : \Omega \to \bigcup_z M_z$$

is meromorphic, if

i) $\varphi(z) \in M_z \qquad \forall z \in \Omega$

ii) $\forall z_0$ $\exists$ nbhd ω of z_0 and $f, g \in A(\Omega)$ s.t.

$$\varphi(z) = f_z / g_z \quad , \text{ for } z \in \omega$$

The set of meromorphic functions is denoted by $M(\Omega)$.

If $F \in A(\Omega)$, then $z \mapsto F_z$ is clearly meromorphic. If $F, G \in A(\Omega)$ and $F \neq G$ then $\exists z \in \Omega$ s.t. $F(z) \neq G(z) \Rightarrow$ $F_z \neq G_z$. Thus we may interpret that

$$A(\Omega) \subset M(\Omega).$$

<u>Lemma</u> $M(\Omega)$ is a ring. If $\varphi \in M(\Omega)$ is s.t. $\varphi|_\omega \neq 0$ for any component ω of Ω, then φ^{-1} exist and $\varphi^{-1} \in M(\Omega)$.

<u>Proof</u> Easy ex. □

If $q \in M_\zeta$, $\zeta \in \Omega$ we are going to assign a value $q(\zeta)$. For this choose ω and $f, g \in A(\Omega)$ s.t. $\zeta \in \omega$ s.t.

$$q = f_\zeta / g_\zeta \quad , \quad g_\zeta \neq 0.$$

Note that $g_\zeta \neq 0$ does not imply $g(\zeta) \neq 0$.

If

$$q = 0 \quad \text{we set} \quad q(\zeta) = 0.$$

If $q \neq 0$ we get from Cor 1.2.10 that

$$f(z) = (z-\zeta)^n f_1(z) \quad , \quad f_1(\zeta) \neq 0$$
$$g(z) = (z-\zeta)^m g_1(z) \quad , \quad g_1(\zeta) \neq 0$$

Clearly $n-m$ and $f_1(\zeta)/g_1(\zeta)$ depend only on q (not on f or g). We define

$$q(\zeta) = \begin{cases} \infty & , \text{ if } n < m \\ f_1(\zeta)/g_1(\zeta) & \text{ if } n = m \\ 0 & , \text{ if } n > m. \end{cases}$$

Thus if $\varphi \in M(\Omega)$ we obtain a map

$$z \longrightarrow \varphi_z(z) = F(z) \in \dot{\mathbb{C}} := \mathbb{C} \cup \{\infty\}.$$

Since an analytic function can vanish only on discrete subset of Ω we obtain a discrete subset D of Ω s.t.

$$(*) \begin{cases} F \text{ is analytic in } \Omega \setminus D \text{ and} \\ 1/F \text{ is analytic in the nbhd of } D. \end{cases}$$

Conversely if F sat. $(*)$ we may define

$$\varphi(z) = F_z \quad , \text{ if } z \notin D$$

$$\varphi(z) = \frac{1}{(1/F)_z} \quad , \text{ if } z \in D$$

and $\varphi \in M(\Omega)$ and $\varphi_z(z) = F(z)$, $\forall z \in \Omega$.

$$F \longleftrightarrow \varphi \quad \text{is one to one}$$

In the sequel we don't distinguish between F and φ.

Remark We say that f is analytic (meromorphic) near z, if there is an open neighborhood ω of z so that f is analytic (meromorphic) in ω. Similarly with compact sets.

1.4.2 Theorem If F is meromorphic in $\Omega \subset \mathbb{C}$ and $\zeta \in \Omega$, then near ζ

$$F(z) = \sum_1^n A_k (z-\zeta)^{-k} + G(z),$$

where A_k are constants and G analytic.
The represent. is unique. If $F_\zeta \neq 0 \Rightarrow$

$$F(z) = (z-\zeta)^n G(z) \qquad \text{(repr. unique)}$$

where $G(\zeta) \neq 0$.

Proof Since

$$F(z) = \frac{f(z)}{g(z)}$$

where f and g are analytic, and by The 1.?.

$$f(z) = (z-\zeta)^k f_1(z) \quad , \quad f_1(\zeta) \neq 0$$

$$g(z) = (z-\zeta)^m g_1(z) \quad , \quad g_1(\zeta) \neq 0$$

the claims follow. □

1.4.3 Theorem (Mittag-Leffler) Let
$A = \{z_j \mid j \in \mathbb{N}\} \subset \Omega$ be a discrete set and
f_j meromorphic near z_j. Then $\exists\, f : \Omega \to \mathbb{C}$
meromorphic s.th. $f : \Omega \setminus A \to \mathbb{C}$ is analytic
and $f - f_j$ is analytic near z_j.

<u>Proof</u> Th. 1.4.2 $\Rightarrow$ We may assume

$$f_j = \sum_1^{n_j} A_{jk}(z-z_j)^{-k}$$

We seek f in the form

$$f(z) = \sum_1^{\infty}(f_j(z) - u_j(z))$$

where $u_j \in A(\Omega)$. Choose $K_j \Subset \Omega$

$K_j \subset K_{j+1}$, $\hat{K}_j = K_j$ s.t. $\bigcup K_j = \Omega$.

We may assume

$$z_k \notin K_j \text{, when } k \geq j \text{,}$$

since A has no accumulation points in Ω.

By Runge, $\exists\ u_j \in A(\Omega)$ s.t.

$$|f_j(z) - u_j(z)| < 2^{-j} \quad \text{in } K_j.$$

This implies that

$$g_k(z) = \sum_k^{\infty}(f_j(z) - u_j(z))$$

converg unif. on K_k. Since each f_j is analytic near K_k, for $j \geq k$ we see that g_k is analytic in $\mathring{K}_k$, where $\mathring{K}_k$ is interior of K_k.

Hence f is well defined

$$f = \underbrace{\sum_{j=1}^{k-1} f_j - u_j}_{\substack{\text{has poles} \\ \text{in } z_1, \dots, z_{k-1}}} + \underbrace{\sum_{j=k}^{\infty} f_j - u_j}_{\text{is analytic in } \mathring{K}_j}$$

Clearly also $f - f_j$ is analytic near z_j.

$\square$

<u>Theorem 1.4.3'</u> (Mittag-Leffler). Assume $\Omega = \cup \Omega_j$, $\Omega_j \subset \mathbb{C}$. If $f_j \in M(\Omega_j)$ and $f_j - f_k \in A(\Omega_j \cap \Omega_k)$ $\forall j, k$, then $\exists f \in M(\Omega)$ s.t. $f - f_j \in A(\Omega_j)$, $\forall j$.

<u>Proof</u> Ex. $\square$

<u>1.4.4 Theorem</u> $\forall f \in C^\infty(\Omega)$

$$\bar\partial u = f$$

has a sol. $u \in C^\infty(\Omega)$.

<u>Proof</u> Choose K_j as in the proof of Thm 1.4.3

Take $\psi \in C_0^\infty(\Omega)$ s.t.

$$\psi_j|_{K_j} \equiv 1 \quad .$$

Finally set

$$\varphi_1 = \psi_1 \qquad \text{and}$$

$$\varphi_j = \psi_j - \psi_{j-1} \quad , \quad j > 1 .$$

Then

$$\varphi_j = 0 \quad \text{near } K_{j-1} \quad \text{and}$$

$$\sum \varphi_j = 1 \quad \text{in } \Omega$$

Note

$$\sum_{j=1}^{k} \varphi_j(z) = \psi_k(z) = 1$$

for k large enough that $z \in K_k$.

By Theorem 1.2.2 $\exists$ $u_j \in C^\infty(\mathbb{R}^2)$ s.t.

$$\bar\partial u_j = \varphi_j f$$

Hence

$$\bar\partial u_j = 0 \qquad \text{near } K_{j-1} .$$

Hence Runge $\Rightarrow$ $\exists$ $v_j \in A(\Omega)$ s.t.

$$|u_j - v_j| \leq 2^{-j} \quad \text{in } K_{j-1}$$

The sum

$$u = \sum_1^\infty u_j - v_j \qquad \text{conv. unif on compact}$$

subsets of Ω