<u>1.2.6 Corollary</u> (Stieltjes - Vitaly). Assume $u_n \in A(\Omega)$ and $\forall\ K \Subset \Omega$

$$\sup_n \sup_{z \in K} |u_n(z)| \leq C < \infty .$$

Then $\exists$ subseq u_{n_j} and $u \in A(\Omega)$ s.t.

$$u_{n_j} \to u$$

unif on compacts.

<u>Proof</u> As in the proof of Cor 1.2.5 we get

$$\sup_{z \in K} |\nabla u_n(z)| \leq C < \infty$$

where C is indep. of n. Thus by the mean value the family $u_n(z)$ is unif bounded and equicont. on every compact set. Hence the claim follows from Arzeli's theorem and Cor 1.2.5.

$\square$

<u>1.2.7 Cor</u> Assume the power series

$$u(z) = \sum_0^\infty a_n z^n$$

converges in the disc $\underset{D(0,\rho)}{\overset{}{W}}$. Then $u \in A(W$

Proof The series conv. unif on every smaller disc. Thus the claim follows from Cor. 1.2.5.

$\square$

1.2.8 Theorem If u is analytic in $\Omega = \{|z| < r\}$, then

$$u(z) = \sum_0^\infty \frac{u^{(n)}(0)}{n!} z^n$$

with unif convergence on K, if $K \Subset \Omega$.

Proof Let $0 < r_1 < r_2 < r$. By (1.2.3) ($\Leftarrow$ Theorem 1.2.1) we have

$$(1.2.6) \qquad u(z) = \frac{1}{2\pi i} \int_{|\zeta| = r_2} \frac{u(\zeta)}{\zeta - z} \, d\zeta \;, \qquad |z| < r_1$$

We have for $|z| \le r_1$ and $|\zeta| = r_2$

$$(\zeta - z)^{-1} = \frac{1}{1 - z/\zeta} \frac{1}{\zeta} = \sum_0^\infty \frac{z^n}{\zeta^{n+1}}$$

and the above series is unif. and abs. conv.

Thus we have

$$(*) \qquad u(z) = \sum_0^\infty a_n z^n \;, \quad |z| \le r_1$$

where $a_n = \frac{1}{2\pi i} \int_{|\zeta| = r_2} \zeta^{-n-1} u(\zeta) \, d\zeta$,

and $(*)$ is unif. conv. in $|z| < r_1$.

Finally (*) $\Rightarrow$

$$u^{(n)}(0) = n!\, a_n = \frac{n!}{2\pi i}\int_{|\xi| = r_n} \frac{u(\xi)}{\xi^{n+1}}\, d\xi .$$

$\square$

1.2.9 Corollary (The uniqueness of analytic continuation)
If $u \in A(\Omega)$ and Ω connected, and $\exists z \in \Omega$ s.t.

$$(1.2.7) \qquad u^{(k)}(z) = 0 \qquad \forall\ k = 0, 1, \dots$$

Then $u = 0$ in Ω.

Proof Set

$$A = \{ z \in \Omega \mid (1.2.7) \text{ holds for } z \}.$$

Then A is closed since

$$A = \bigcap_{k=0}^{\infty} A_k$$

where A_k are closed sets. By Theorem 1.2.8 the set A is open.

$\square$

1.2.10 Corollary If $\Omega = \{ z : |z| < r \}$ and $0 \neq u : \Omega \to \mathbb{C}$ analytic, then $\exists!\, n \in \mathbb{N}_0$, $\exists!\, v \in A(\Omega)$, $v(0) \neq 0$ s.t.

$$u(z) = z^n v(z)$$

Proof By Thm 1.2.8

$$u(z) = \sum_0^\infty \frac{u^{(k)}(0)\, z^k}{k!}$$

Let $n = \max\{ l \mid u^{(k)}(0) = 0 ,\ \text{for } k < l \}$

Then $$u(z) = \sum_{k \geq n} \frac{u^{(k)}(0)\, z^k}{k!}$$

$$= z^n \underbrace{\left(\frac{u^{(n)}(0)}{n!} + \frac{u^{(n+1)}}{(n+1)!} z + \ldots \right)}_{v(z)}$$

and $$v(0) = \frac{u^{(n)}(0)}{n!} \neq 0 .$$

□

1.2.11 Theorem Let $u \in A(\Omega)$, $\Omega = \{ z \mid |z - z_0| < r \}$.

If $|u(z)| \leq |u(z_0)|$ for $z \in \Omega$, then u is const.

Proof Assume $u(z_0) \neq 0$. We have for $0 < \rho < r$

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho e^{i\theta})\, d\theta$$

$\Rightarrow$

$$\int_0^{2\pi} \left(1 - \frac{u(z_0 + \rho e^{i\theta})}{u(z_0)} \right) d\theta = 0$$

$$\Rightarrow \int_0^{2\pi} \underbrace{\mathrm{Re}\left(1 - \frac{u(z_0 + \rho e^{i\theta})}{u(z_0)} \right)}_{\geq 0} d\theta = 0$$

$$\Rightarrow \frac{u(z_0 + \rho e^{i\theta})}{u(z_0)} = 1 \qquad \forall \rho, \theta.$$

□

1.2.13 <u>Corollary</u> (Maximum principle) Let Ω be bounded, $u \in C(\overline{\Omega}) \cap A(\Omega)$. Then max of $|u|$ in $\overline{\Omega}$ is max of $|u|$ on $\partial\Omega$.

Proof Assume max is attained in $z_0 \in \Omega$.

. Theorem 1.2.12 $\Rightarrow$ u is const. in the component containing z_0. $\square$

1.3 The Runge approximation theorem

(Theorem 1.2.8 $\Rightarrow$ $u \in A(\mathbb{C})$ can be approximated unif. by polynomials on K, if $K \Subset \Omega$.

<u>1.3.1 Theorem</u> Let $K \Subset \Omega \Subset \mathbb{C}$. Then the FAE

a) If u is analytic in the nbhd of K, then u can be approximated unif. in K by functions in $A(\Omega)$

b) $\Omega \setminus K$ has no rel. compact component in Ω.

c) $\forall z \in \Omega \setminus K$ $\exists f \in A(\Omega)$ s.t.

$$|f(z)| > \sup_K |f| .$$

1.3.2 Corollary $\forall u$ analytic in the nbhd of $K \Subset \mathbb{C}$ can be approximated unif by polynomials $\Leftrightarrow$

$\mathbb{C} \setminus K$ is connected $\Leftrightarrow$

$\forall \quad z \in \mathbb{C} \setminus K \quad \exists$ polyn. p s.t.

$$|p(z)| > \sup_K |p|$$

Proof Take $\Omega = \mathbb{C}$ in Thrm 1.3.1 and use $(*)$. $\square$

Proof of Thrm 1.3.1 : $c) \Rightarrow b)$: Assume $b)$ is not true. Then $\exists$ a component $\mathcal{O}$ of $\Omega \setminus K$ s.t. $\overline{\mathcal{O}}$ is comp and $\subset \Omega$. Then

$$\partial \overline{\mathcal{O}} = \partial \mathcal{O} \subset K$$

and max. principle $\Rightarrow$

$$(1.3.2) \qquad \sup_{\overline{\mathcal{O}}} |f| \leq \sup_K |f|$$

which contradicts with c)

$a) \Rightarrow b)$: Assume $b)$ is not valid. If a) would be valid, $\forall f$ analytic in nbhd of K, $\exists f_n \in A(\Omega)$ s.t.

s.t.

$$\sup_K |f_n - f| \to 0$$

By applying (1.3.2) we get

$$\sup_{\bar{\mathcal{O}}} |f_n - f_m| \to 0$$

$\Rightarrow$ $f_n \to F$ unif in $\bar{\mathcal{O}}$ and

$F|_K = f \Rightarrow F|_{\partial\mathcal{O}} = f|_{\partial\mathcal{O}}$

Finally $F \in A(\mathcal{O}) \cap C(\bar{\mathcal{O}})$.

We will choose $f(z) = \frac{1}{z-\zeta}$, $\zeta \in \mathcal{O}$.

$\Rightarrow$ $F(z)(z-\zeta) = 1$ on $\partial\mathcal{O}$

$\Rightarrow$ $F(z)(z-\zeta) = 1$ in $\mathcal{O}$ (Max. princ.)

Take $z = \zeta$ to obtain a contradiction

b) $\Rightarrow$ a): Let μ be a measure on K s.t.

(1) $\mu \perp A(\Omega) \Leftrightarrow \int_K f \, d\mu = 0 \quad \forall f \in A(\Omega)$

Assume we can show that (1) implies

(2) $\mu \perp A(\Omega_K) \Leftrightarrow \int_K f \, d\mu = 0 \quad \forall f \in A(\Omega_K)$

where Ω_K is any open nbhd of K in Ω.

Then denote by C_0 and C_1 two subspace of $C(K) = \{ f : K \to \mathbb{C} \mid f \text{ cont.} \}$

$$C_0 = \overline{\{ f \in A(\Omega) \}}$$

$$C_1 = \overline{\{ f \in A(\Omega_K) \mid \text{for some } \Omega_K \}}$$

where the closures are taken in $C(K)$.

By Riesz-theorem and by Hahn-Banach the claim (2) is equivalent to $C_0 = C_1$.

Hence it is enough to show (2).

For this set

$$\varphi(\zeta) = \int \frac{d\mu(z)}{\zeta - z}, \qquad \zeta \in \mathbb{C} \setminus K.$$

Theorem 1.2.2 $\Rightarrow$ φ is analytic in $\mathbb{C} \setminus K$ and

$$\varphi^{(k)}(\zeta) = k! \int \frac{1}{(z-\zeta)^{k+1}} \, d\mu(z), \text{ if } \zeta \in \mathbb{C} \setminus K.$$

By (1)

$$\varphi^{(k)}(\zeta) = 0, \qquad \forall k, \forall \zeta \in \mathbb{C} \setminus \Omega)$$

$\Rightarrow$ $\varphi = 0$ in every comp. of $\mathbb{C} \setminus K = K^c$ which intersects $\Omega^c = \mathbb{C} \setminus \Omega$. Since

$$\int z^n \, d\mu(z) = 0$$

we get from $\frac{1}{\zeta - z} = \sum_0^\infty z^n \zeta^{-n-1}$

that φ vanishes also in the unbounded comp. of K^c. By b) we have

(3) $$\varphi = 0 \quad \text{in } K^c .$$

To this end assume $f \in A(\omega)$ and ω is nbhd of K. Choose $\psi \in C_0^\infty(\omega)$, $\psi \equiv 1$ in a nbhd of K. Then for $z \in K$

$$f(z) = \psi(z) f(z) = \frac{1}{2\pi i} \int \frac{f(\zeta)\, \bar\partial \psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar\zeta .$$

Now $\bar\partial \psi = 0$ in a nbhd of K and we get

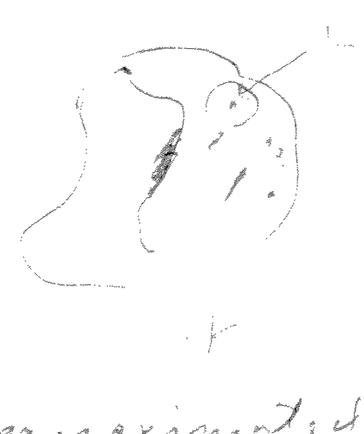
$$\int f(z)\, d\mu(z) = \frac{1}{2\pi i} \int f(\zeta)\, \bar\partial \psi(\zeta)\, \varphi(\zeta)\, d\zeta \wedge d\bar\zeta$$

$= 0$ since φ vanishes in K^c.

Thus a) $\Leftrightarrow$ b). We are left to show

b) $\Rightarrow$ c): Assume b) and $z \in \Omega \setminus K$. Choose a closed disc L s.t. $L \subset \Omega \setminus K$.

Then also $K \cup L$ also sat b). Since b) $\Rightarrow$ a) a function that is 0 in the nbhd of K and 1 — " — of L can be approximated



uniformly by functions in $A(\Omega)$ $\Rightarrow$

$\exists\ f \in A(\Omega)$ s.t.

$$|f(z)| < 1/2 \quad \text{in } K$$

$$|f(z) - 1| < 1/2 \quad \text{in } L .$$

This proves c) $\boxed{}$

Assume $K \Subset \Omega$. We define

$$\hat{K} = \hat{K}_\Omega = \{ z \in \Omega \mid |f(z)| \le \sup_K |f| , \ \forall f \in A(\Omega) \}$$

and call $\hat{K}$ $A(\Omega)$-hull of K.

<u>Lemma</u> i) $d(K, \Omega^c) = d(\hat{K}, \Omega^c)$

(ii) $\hat{K} \subset ch(K)$

where $ch(K)$ is the convex hull of K

<u>Proof</u> i) Assume $\zeta \in \Omega^c = \mathbb{C} \setminus \Omega$ and let $f(z) = \frac{1}{z - \zeta}$. Then $f \in A(\Omega)$ and $z \in \hat{K}$ $\Rightarrow$

$$\frac{1}{|z - \zeta|} \le \sup_{w \in K} \frac{1}{|w - \zeta|} = \frac{1}{d(\zeta, K)}$$

Thus

$$\frac{1}{d(\hat{K}, \zeta)} \le \frac{1}{d(\zeta, K)}$$

and since $K \subset \hat{K}$ we have

$$d(\zeta, K) = d(\zeta, \hat{K}) \qquad \forall \zeta \in \Omega^c$$

(ii) Take $\alpha \in \mathbb{C}$ and

$$f(z) = e^{\alpha z}$$

Then

$$|e^{\alpha z}| \leq \sup_{w \in K} |e^{\alpha w}| \qquad , \ \forall z \in \hat{K}.$$

$$\Rightarrow \quad \mathrm{Re}\, \alpha z \leq \sup_{w \in K} \mathrm{Re}\, \alpha w$$

By changing to real notations $z = (x_1, x_2) \in \hat{K}$ we get

$$\theta \cdot x \leq \sup_{w \in K} \theta \cdot w \qquad , \ w = (w_1, w_2)$$

$\forall \theta \in S^1$, $\forall x \in \hat{K}$. Thus

$$\hat{K} \subset ch(K).$$

]

<u>Lemma 2</u> i) $\hat{\hat{K}} = K$

(ii) The hypothesis of Runge's theorem (i.e. (ii)) holds for $\hat{K}$, $\hat{K}$ is compact and

$$K \subset \hat{K} \subset \Omega$$

Proof (i)

(1) $z \in \hat{\hat{K}}_\Omega \iff |f(z)| \le \sup_{z \in \hat{K}} |f(z)|$

But $z \in \hat{K} \iff |f(z)| \le \sup_{z \in K} |f(z)|$

Thus $\sup_{z \in \hat{K}} |f(z)| \le \sup_{z \in K} |f(z)|$

Since $K \subset \hat{K}$ we have thus

$$\sup_{z \in \hat{K}} |f(z)| = \sup_{z \in K} |f(z)| .$$

This and (1) prove $\hat{\hat{K}} = \hat{K}$.

ii) Clearly $\hat{K}$ is closed and since the convex hull of a compact set is bounded we get from Lemma 1 (ii) that $\hat{K}$ is also bounded. Thus $\hat{K}$ is compact. By 4 of Theorem 1.3.1 the claims a), b), and c) hold for $\hat{K}$.

$\square$

Lemma 3 $\exists\ K_j \Subset \Omega$ s.t. $K_j \subset K_{j+1}$

$\hat{K}_j = K_j \quad , \quad j = 1, 2, \ldots$

$L \Subset \Omega \Rightarrow \exists\ j$ s.t. $L \subset K_j$

Proof Ex. $\square$

Remark Lemma 3 $\Rightarrow$ $\Omega = \bigcup_j K_j$

1.3.3 Theorem $\hat{K}_\Omega$ is the union of K and components of $\Omega \setminus K$ that are rel. comp. in Ω.

Proof Let O be a comp. of $\Omega \setminus K$ s.t. $\bar{O} \Subset \Omega$.

By (1.3.2) i.e.

$$\sup_{\bar{O}} |f| \leq \sup_K |f| \quad , \quad f \in A(\Omega)$$

we have $\bar{O} \subset \hat{K}$.

Let K_1 be the union of K and all such $\bar{O}$:s

Now $K_1 \subset \hat{K}$ and $\Omega \setminus K_1$ is open since $\Omega \setminus \bar{O}$ is open. Thus K_1 is compact.

By definition of K_1 no component of $\Omega \setminus K_1$ is rel. comp. in Ω. Thus by Theorem 1.3.2 c) there holds for K_1 i.e.

$$z \in \Omega \setminus K_1 \Rightarrow z \in \Omega \setminus \hat{K} \quad \text{i.e.}$$

$$\hat{K} \subset K_1$$

$\square$

<u>Theorem 1.3.4</u> Let $\Omega_1 \Subset \Omega_2 \Subset \mathbb{C}$. TFAE

a) If $f \in A(\Omega_1)$, and $K \Subset \Omega_1$ then f can be approximated unif. on K by $A(\Omega_2)$.

b) If $\Omega_2 \setminus \Omega_1 = L \dot{\cup} F$, L is comp. and F closed in Ω_2, then $L = \emptyset$.

c1) $K \Subset \Omega_1 \Rightarrow \hat{K}_{\Omega_2} = \hat{K}_{\Omega_1}$

c2) $K \Subset \Omega_1 \Rightarrow \hat{K}_{\Omega_2} \cap \Omega_1 = \hat{K}_{\Omega_1}$

c3) $K \Subset \Omega_1 \Rightarrow \hat{K}_{\Omega_2} \cap \Omega_1$ is compact.

<u>Proof</u> The property c) in Thm 1.3.1 says that

$$K = \hat{K}_{\Omega}.$$

Always clear

$$\hat{K}_{\Omega_1} \subset \Omega_1 \cap \hat{K}_{\Omega_2}$$

If a) holds, $z \in \Omega_1 \cap \hat{K}_{\Omega_2}$ and $f \in A(\Omega_1)$, then $\exists f_j \in A(\Omega_2)$ s.t.

$$f_j \to f \quad \text{unif on } \{z\} \cup K.$$

Thus

$$|f(z)| \le \sup_K |f|, \quad \forall f \in A_{\Omega_1}$$

We have thus shown a) $\Rightarrow$ c$_2$). The claim c$_2$) $\Rightarrow$ c$_3$) is obvious.

c$_3$) $\Rightarrow$ a) : Set $K' = \hat{K}_{\Omega_2} \cap \Omega_1$ and $K'' = \hat{K}_{\Omega_2} \cap \Omega_1^c$. By c$_3$) K' is compact and, since Ω_1^c is closed in $\hat{K}_{\Omega_2}$, K'' is also compact. Note

$$K_{\Omega_2} = K' \cup K'' \quad \text{and} \quad K' \cap K'' = \emptyset .$$

Now if $f \in A(\Omega_1)$, then a function

$$g : K_{\Omega_2} \to \mathbb{C} ,$$

$$g|_{K'} = f \; , \; g|_{K''} = 1$$

can be approximated on $K_{\Omega_2} = K' \cup K''$ by $A(\Omega_2)$. This proves a). Moreover setting $f = 0 \Rightarrow z \in \hat{K}_{\Omega_2} = K' \cup K''$

$$|f(z)| \le \sup_{z \in K \subset K'} |f| = 0$$

This is possible only if $K'' = \emptyset$. Thus $\hat{K}_{\Omega_2} = K' = \hat{K}_{\Omega_2} \cap \Omega_1$.

Since a) also implies c$_2$) we get

$$\hat{K}_{\Omega_2} = \hat{K}_{\Omega_1} \quad \text{i.e. } c_1) .$$

We have so far shown that a), c1), c2) and c3) are equivalent.

Finally, it needs to be shown that c1) ⇔ b).

Choose ω be open in Ω_2 s.t. $L \subset \omega$ and $\overline{\omega} \cap F = \emptyset$ (and $\overline{\omega}$ rel comp. in Ω_2; recall L comp., F closed, $L \cap F = \emptyset$.)

Now

$$(1) \qquad \partial\omega \cap (\Omega_2 \setminus \Omega_1) = \emptyset$$

since $\partial\omega \cap F = \emptyset$ and $\partial\omega \cap L = \emptyset$. Also

$$(2) \qquad \partial\omega \subset \Omega_2 \quad \text{since } \omega \subset \Omega_2 .$$

By (1) and (2)

$$\partial\omega \subset \Omega_1 .$$

Let $K = \partial\omega$. By maximum principle

$$L \subset \omega \subset \hat{K}_{\Omega_2} .$$

By c1) we have

$$L \subset \hat{K}_{\Omega_1} \subset \Omega_1 \implies L = \emptyset .$$

b) ⇒ c1):

Let O be a component of $\Omega_2 \setminus K$ s.t. $\overline{O}$ is comp. in Ω_2.

Then

$$\partial \mathcal{O} \subset K \subset \Omega_1 \quad \Rightarrow$$

$$L := \overline{\mathcal{O}} \cap (\Omega_2 \setminus \Omega_1) = \mathcal{O} \cap (\Omega_2 \setminus \Omega_1)$$

is comp. in $\mathcal{O}$

$$L \Subset \mathcal{O}.$$

Also

$$\mathcal{O}^c \cap (\Omega_2 \setminus \Omega_1) \quad \text{is closed in } \Omega_2$$

since $\partial \mathcal{O} \subset \Omega_1$. Hence e) implies that $L = \overline{\mathcal{O}} \cap (\Omega_2 \cap \Omega_1) = \emptyset \Rightarrow \mathcal{O} \subset \Omega_1$.

Theorem 1.3.3 $\Rightarrow$

$$\hat{K}_{\Omega_2} \subset \hat{K}_{\Omega_1}$$

Since trivially $\hat{K}_{\Omega_1} \subset \hat{K}_{\Omega_2}$ the cond (1) follows

1.4 THE MITTAG-LEFFLER THEOREM

We will define *meromorphic* functions in the way that generalises easily to several variables.