COMPLEX ANALYSIS IN SEVERAL VARIABLES

LUENNOT SYKSY 2009

LASSI PÄIVÄRINTA

OPPIKIRJA: LARS HÖRMANDER, AN INTRODUCTION TO COMPLEX ANALYSIS IN SEVERAL VARIABLES

1. ANALYTIC FUNCTIONS IN THE COMPLEX PLANE

1.1. INTRODUCTION

Let $u \in C^1(\Omega)$, $\Omega \subset \mathbb{C} = \mathbb{R}^2$ open set.

$$z = x + iy, \quad \bar{z} = x - iy, \quad x, y \in \mathbb{R}$$

Thus

$$du = \partial_x u\, dx + \partial_y u\, dy = \partial u\, dz + \bar{\partial} u\, d\bar{z}$$

where

$$\partial u = \tfrac{1}{2}(\partial_x - i\partial_y) \quad \text{and} \quad \bar{\partial} u = \tfrac{1}{2}(\partial_x + i\partial_y).$$

Of course

$$dz = dx + i\,dy \quad \text{and} \quad d\bar{z} = dx - i\,dy.$$

<u>1.1.1 Def</u> $u \in C^1(\Omega)$ is analytic (holomorphic) if $\bar{\partial} u = 0$. The set of all analytic functions is denoted $A(\Omega)$.

<u>Remark</u> If $u \in A(\Omega)$, then we denote

$$\partial u = \frac{\partial u}{\partial z} = u' \quad \text{and hence}$$

$$du = u'\,dz$$

<u>Ex</u> (1) $\bar\partial z^n = 0 \Rightarrow \sum_{k=0}^{n} a_k z^k$ is analytic

$$d(z^n) = n z^{n-1}\,dz$$

(2) Define $e^z = e^x(\cos y + i \sin y)$

Simple calculations of real variables reveals

$$de^z = e^z\,dz$$

and hence e^z is analytic

<u>Lemma</u> $A(\Omega)$ is an algebra.

<u>Proof</u> Let $u, v \in A(\Omega)$, then for $\alpha, \beta \in \mathbb{C}$

$$\bar\partial(\alpha u + \beta v) = \alpha \bar\partial u + \beta \bar\partial v = 0$$

Next

$$d uv = u\,dv + v\,du$$

yields

$$\bar\partial uv = u \bar\partial v + v \bar\partial u = 0$$

Hence $uv \in A(\Omega)$. $\square$

$u \in A(\Omega) \Rightarrow u^{-1} \in A(\mathrm{Dom}\,u)$?

Denote $e_1 = dx$ and $e_2 = dy$ and consider the complex plane of differentials

$$\tilde{\mathbb{C}} = \{ s e_1 + i t e_2 \mid s, t \in \mathbb{R} \}.$$

In this space the map

$$dz \mapsto du = u' dz$$

is a rotation followed by multiplication with $|u'|$. Hence the Jacobian of

$$u : \Omega \to \mathbb{R}, \quad \Omega \subset \mathbb{R}^2$$

is

$$J_u = |u'(z)|^2 .$$

This follows also by direct calculation

<u>Lemma</u> For $u \in C^1(\Omega)$ it holds

$$J_u := \begin{vmatrix} \partial_x u_1 & \partial_x u_2 \\ \partial_y u_1 & \partial_y u_2 \end{vmatrix} = \begin{vmatrix} \partial u & \partial \bar{u} \\ \bar{\partial} u & \bar{\partial} \bar{u} \end{vmatrix} = |\partial u|^2 - |\bar{\partial} u|^2$$

Especially if $u \in A(\Omega)$, we have

$$J_u = |\partial u|^2 .$$

<u>Proof</u> Ex.

Implicit function theorem yields:

$u'(z_0) \neq 0 \Rightarrow$

u is homeo in nbhood of z_0 and if $w_0 = u(z_0)$ the inverse of u $z(w)$ is C^1 in the nbhood of w_0

Moreover the inverse of u denoted by $z(w)$ satisfies

$$u(z(w)) = w$$

Chain rule
$$u'(z(w))\, dz = dw$$

Hence $w \mapsto z(w)$ is analytic and

$$\frac{\partial z}{\partial w} = z'(w) = \frac{1}{u'(z(w))} .$$

<u>1.2 Cauchy's formula</u> Let $\omega \subset \mathbb{C}$ bounded and open with piecewise C^1-boundary.
If $u \in C^1(\bar{\omega})$, then Stokes theorem says that

(1.2.1) $$\int_{\partial\omega} u\, dz = \int_{\omega} du \wedge dz$$

<u>Proof</u> (by using Gauss). From differential forms we will need only the following facts

(1) $dx \wedge dx = 0$, $dy \wedge dy = 0$, $dx \wedge dy = -dy \wedge dx$

and

(2) $$\int_\omega u \, dx \wedge dy = \int_\omega u \, dm ,$$

where $\int_\omega u \, dm$ is the normal Lebesgue-integral in $\mathbb{R}^2$.

Note first that

(3) $$d\bar{z} \wedge dz = (dx - i dy) \wedge (dx + i dy)$$
$$\overset{(1)}{=} -i dy \wedge dx + i dx \wedge dy = 2i dx \wedge dy$$

Since

$$du \wedge dz = (\bar{\partial} u \, d\bar{z} + \partial u \, dz) \wedge dz = \bar{\partial} u \, d\bar{z} \wedge dz$$

we need to show

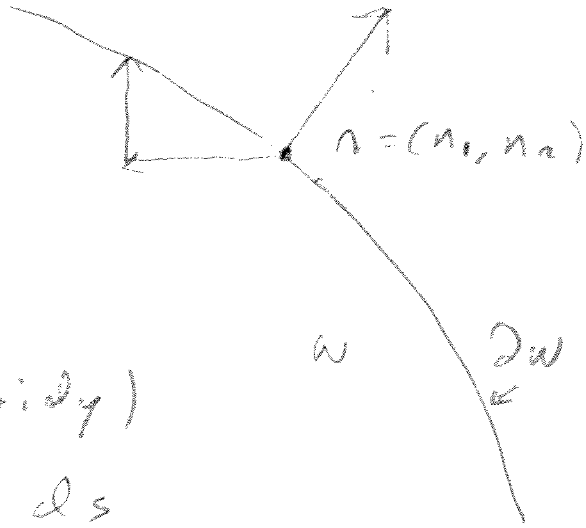
(1.2.2) $$\int_{\partial\omega} u \, dz = 2i \int_\omega \bar{\partial} u \, dx \wedge dy = \int_\omega \bar{\partial} u \, d\bar{z} \wedge dz$$

The last equality holds by (3). We show

(4) $$\mathrm{Re} \int_{\partial\omega} u \, dz \;\; 2 \mathrm{Re} \left(i \int_\omega \bar{\partial} u \, dm \right) = -2 \, \mathrm{Im} \int_\omega \bar{\partial} u \, dm.$$

The claim for imaginary part is an exercise.

Geometry $\Rightarrow$



$$(5)\quad \begin{cases} dx = -n_2\, ds \\ dy = n_1\, ds \end{cases}$$

Now

$$\operatorname{Re} u\, dz = \operatorname{Re}(u_1 + i u_2)(dx + i\, dy)$$

$$= u_1\, dx - u_2\, dy \overset{(5)}{=} -F\cdot n\, ds$$

where $F = (u_2, u_1)$.

By Gauss divergence theorem

$$(6)\quad \operatorname{Re}\int_{\partial\omega} u\, dz = -\int_{\partial\omega} F\cdot n\, ds = -\int_{\omega} \nabla\cdot F\, dm$$

$$= -\int_{\omega} \nabla\cdot F\, dx\wedge dy$$

But

$$2\operatorname{Im}\bar\partial u = \operatorname{Im}(\partial_x + i\partial_y)(u_1 + i u_2)$$

$$= \partial_y u_1 + \partial_x u_2 = \nabla\cdot F$$

Thus

$$2\operatorname{Im}\int_{\omega}\bar\partial u\, dm = \int \nabla\cdot F\, dm$$

$$\overset{(6)}{=} -\operatorname{Re}\int_{\partial\omega} u\, dz$$

proving (4). $\square$

Also $\partial\omega$ is oriented so that ω lies left of $\partial\omega$.

<u>Corollary</u> If $u \in C^1(\bar\Omega)$ is analytic then

$$\int_{\partial\omega} u\, dz = 0$$

<u>Proof</u> Formula 1.2.2. $\square$

<u>1.2.1 Theorem</u> If $u \in C^1(\bar\Omega)$, we have for $\zeta \in \omega$

$$(1.2.3) \qquad u(\zeta) = \frac{1}{2\pi i} \int_{\partial\omega} \frac{u(z)}{z-\zeta}\, dz + \int_{\omega} \frac{\bar\partial u}{z-\zeta}\, dz \wedge d\bar z$$

Especially if u is analytic

$$(7) \qquad u(\zeta) = \frac{1}{2\pi i} \int_{\partial\omega} \frac{u(z)}{z-\zeta}\, dz$$

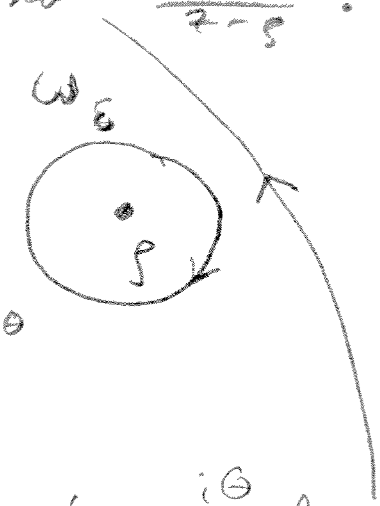
<u>Remark</u> The formulas (1.2.3) and (7) are called Cauchy's integral formula.

Proof Put

$$\omega_\varepsilon = \{ z \in \omega \mid |z-\zeta| > \varepsilon \}$$

where $\varepsilon > 0$ is so small that $\omega_\varepsilon \subset \Omega$.

Since $z \mapsto \frac{1}{z-\rho}$ is analytic in ω_ε we get from (1.2.2) applied to $\frac{u(z)}{z-\rho}$:



$$\int_{\omega_\varepsilon} \frac{\bar\partial u}{z-\rho}\, d\bar z \wedge dz$$

$$= \int_{\partial\omega} \frac{u(z)}{z-\rho}\, dz - \int_0^{2\pi} u(\rho+\varepsilon e^{i\theta})\, i\,d\theta$$

since for $z = \rho + \varepsilon e^{i\theta}$, $dz = i\varepsilon e^{i\theta} d\theta$
$= i(z-\rho)\, d\theta$.

Now since $\frac{\bar\partial u}{z-\rho}$ is integrable over ω and u cont at ρ, we get

(9) $$\lim_{\varepsilon\to 0} \int_0^{2\pi} u(\rho+\varepsilon e^{i\theta})\, i\,d\theta = 2\pi i\, u(\rho)$$

(10) $$\lim_{\varepsilon\to 0} \int_{\omega_\varepsilon} \frac{\bar\partial u}{z-\rho}\, d\bar z \wedge dz = \int_{\omega} \frac{\bar\partial u}{z-\rho}\, d\bar z \wedge dz$$

and the claim follows from (8), (9) and (10). ◻

The converse is true in the following sense

<u>1.2.2 Theorem</u> If μ is a measure with compact support in $\mathbb{C}$, then

$$u(\zeta) = \int \frac{d\mu(z)}{z-\zeta}$$

is analytic and C^∞ outside supp μ. If

$$\mu = \frac{\varphi}{2\pi i}\, dz \wedge d\bar{z} = \frac{1}{\pi}\varphi\, dm \qquad \text{in } \omega \subset \mathbb{C}$$

Lebesgue meas.

where $\varphi \in C^k(\omega)$, then $u \in C^k(\omega)$ and $\bar\partial u = \varphi$, if $k \geq 1$.

<u>Remark</u> The support of μ is smallest closed set K with the property that

$$\int_{\mathbb{C}\setminus K} f\, d\mu = 0 \quad , \text{ for all cont. } f.$$

<u>Proof</u> u is C^∞ outside K since

$$\frac{1}{z-\zeta} \text{ is } C^\infty \quad \text{for } z \in K \text{ and } \zeta \in \mathbb{C}\setminus K.$$

Next

$$\bar\partial_\zeta \frac{1}{z-\zeta} = 0 \qquad \text{if} \qquad \zeta \neq z.$$

Hence if $\zeta \notin$ supp K we have

$$\bar\partial u(\zeta) = \int \bar\partial_\zeta \frac{1}{z-\zeta}\, d\mu(z) = 0 .$$

For the second claim assume $\omega = \mathbb{R}^2$ first.

Now

$$u(\zeta) = \frac{1}{\pi}\int \frac{\varphi(z)}{(z-\zeta)}\, dm(z) = \frac{-1}{\pi}\int \frac{\varphi(\zeta - z)}{z}\, dm(z)$$

$$= -\frac{1}{2\pi i}\int \frac{\varphi(\zeta - z)}{z}\, dz \wedge d\bar z$$

Now $\frac{1}{z}$ is integrable over compact sets and φ is assumed to have a compact support we may diff. under the integration sign k times $\Rightarrow$ $u \in C^k$. Moreover

$$\bar\partial u(\zeta) = \frac{-1}{2\pi i}\int \bar\partial_\zeta \frac{\varphi(\zeta - z)}{z}\, dz \wedge d\bar z$$

$$= \frac{1}{2\pi i}\int \frac{1}{z-\zeta}\,\bar\partial \varphi(z)\, dz \wedge d\bar z$$

Apply Theorem 1.2.1 with $u = \varphi$ and ω a disc s.t. $\operatorname{supp}\varphi \subset \omega$ to get

$$\bar\partial u(\zeta) = \varphi .$$

Finally, if $\omega \neq \mathbb{C}$ and $z_0 \in \omega$ choose a nbhood V of z_0 in ω and $\psi \in C_0^\infty(\omega)$ s.t

$$\psi(z) = 1 \quad \text{for} \quad z \in V.$$

Let

$$\mu_1 = \psi\mu \quad \text{and} \quad \mu_2 = (1-\psi)\mu.$$

Since

$$\mu_1 = \frac{\psi\varphi}{\ldots}\, dz \wedge d\bar{z} \quad , \quad \psi\varphi \in C_0^k(\mathbb{R}^2)$$

Now let

$$u_j(\zeta) = \int \frac{d\mu_j(z)}{z - \zeta} \quad , \; j = 1, 2$$

to get

$$u_1 \in C^k \quad \text{and} \quad \bar{\partial} u_1 = \psi\varphi.$$

Since μ_2 vanishes in V, it follows that $u = u_1 + u_2 \in C^k$ and $\bar{\partial} u = \varphi$ in V. Since V is arbitrary nbhood of z_0 (not touching $\partial\omega$) the same conclusion is true when $\mathbb{C}$ is replaced with ω.

<u>1.2.3 Corollary</u> If $u \in A(\Omega)$ then $u \in C^\infty$ and $u' \in A(\Omega)$

<u>Proof</u> Let ω be a disc with $\bar{\omega} \subset \Omega$.

Then since $\bar\partial u = 0$ we get from (1.2.3)

$$u(\zeta) = \frac{1}{2\pi i}\int_{\partial\omega} \frac{u(z)}{z-\zeta}\,dz = \int \frac{u(z)}{z-\zeta}\,d\mu(z)$$

where $\operatorname{supp}\mu = \partial\omega$. Hence Theorem 1.2.2 $\Rightarrow$ u is C^∞ in ω. Moreover

$$\bar\partial u' = \bar\partial \partial u = \partial \bar\partial u = \partial 0 = 0.$$

$\square$

More precisely

<u>1.2.4 Theorem</u> If $K \Subset \omega \Subset \Omega$, then $\forall j\ \exists C_j > 0$ s.t.

$$\sup_{z\in K} |u^{(j)}(z)| \le C_j \|u\|_{L^1(\omega)}, \quad u \in A(\Omega)$$

<u>Remark</u> Here $K \Subset \omega$ means that K is <u>compact</u> subset of ω and $\omega \Subset \Omega$ means that ω is an <u>open</u> subset of Ω. And $u^{(j)} = (\partial)^j u$.

<u>Proof</u> Choose $\psi \in C_0^\infty(\omega)$ s.t. $\psi \equiv 1$ in the nbhood of K. Now $\bar\partial \psi u = (\bar\partial\psi) u + \psi\bar\partial u = u\,\bar\partial\psi$, since $u \in A(\Omega)$. Theorem 1.2.1 $\Rightarrow$

$$(1.2.3) \qquad \psi(\zeta)u(\zeta) = \frac{1}{2\pi i}\int_\omega \frac{u(z)\,\bar\partial\psi(z)}{z-\zeta}\,dz\wedge d\bar z$$

When $\zeta \in K$ we have $\left|\partial^j \frac{1}{z-\zeta}\right| \le C < \infty$

when $z \in \operatorname{supp} \bar\partial\psi$. Thus for $\zeta \in K$

$$\partial^j u(\zeta) \le \frac{1}{2\pi} \sup_{\substack{\zeta\in K\\ z\in\operatorname{supp}(\psi-1)}} \left|\partial^j \frac{1}{z-\zeta}\right| \int_\omega |u(z)\,\bar\partial\psi(z)|\,dz\wedge d\bar z$$

and the claim follows. □

<u>1.2.5 Corollary</u> If $u_n \in A(\Omega)$ and $u_n \to u$ unif on compacts, then $u \in A(\Omega)$.

<u>Proof</u> Apply (1.2.4) to $u_n - u_m$ to get

$$\sup_{z\in K} |\partial(u_n-u_m)| \le C\,\|u_n-u_m\|_{L^1(\omega)}$$

if $K \Subset \omega \subset \Omega$. Thus ∂u_n conv. unif. on compacts. Moreover since $\bar\partial u_n = 0$ we see that $\bar\partial u_n$ conv. unif on compacts. Hence $u \in C^1$

$$\bar\partial u = \lim_{n\to\infty} \bar\partial u_n = 0.$$

□