Stochastic analysis , Problem sheet n.10. (23.04.08), Spring 2008.

The next lecture is on tuesday 29.4

Exercise 1) Use the definition of conditional expectation to show that if X and Y are independent r.v., then

$$E(f(X,Y)|\sigma(X))(\omega) = E(f(x,Y)) \Big|_{x=X(\omega)}$$

Exercise 2) Let X_t be the strong solution of the stochastic differential X_t.

$$dX_t = X_t b_t dt + f_t dV_t + g_t dW_t$$

where V and W are independent Brownian motions, and b, f and g are deterministic functions in $L^2([0,\infty), dt)$.

i) write down the explicit solution X_t

ii) Show that X_t it is the solution of the SDE

$$dX_t = X_t b_t dt + \sqrt{(f^2 + g^2)} dB_t$$

for some Brownian motion B_t.

iii) Let

$$Y_t = \int_0^t X_s h_s ds + W_t$$

where h_s is a deterministic functions in $L^2([0,\infty), dt)$.

Assume that X_0 is gaussian and that $(X_0, W_t, V_t : t \geq 0)$ are independent r.v.

iv) Show that the process (X_t, Y_t) is jointly gaussian.

Exercise 3) Let $(W_t : t \in [0,T])$ be a brownian motion and a function $h(x) = x^k$ for some $k \in 2\mathbb{N}$. For simplicty you can take $k = 2$ or $k = 4$. Let

$$Z_T = \frac{h(W_T)}{E_P(h(W_T))}$$

and construct the probability measure P_h^T on $\mathcal{F}_T^W$ defined by

$$P_T^h(d\omega) = Z_T(\omega) P_T(d\omega) = \frac{h(W_T)}{E(h(W_T))} P_T(d\omega)$$

i) Use Ito formula to write the Ito representation of the Radon-Nikodym derivative Z_T in the Brownian filtration $\{\mathcal{F}_t^W\}$.

ii) Use Girsanov theorem to compute the $\{\mathcal{F}_t^W\}$-martingale decomposition of W_t under the new measure P^h (that is compute its new drift under).

Exercise 4) Let

$$A_t = \int_0^t W(s) f(s) ds$$

where the determistic function $f(s)$ is continuous on $[0, T]$.

i) Use gaussianity to compute $E(\exp(A_t))$.

ii) Consider on $\mathcal{F}_T^W$ the probability measure

$$P_T^A(d\omega) := \frac{\exp(A_T)}{E_P(\exp(A_T))} P_T(d\omega) = Z_T(\omega) P_T(d\omega)$$

iii) Compute the Ito representation of the likelihood ratio Z_T and use Girsanov theorem to find the drift of the Brownian motion W_t under the measure P^A.

Exercise 5) Combine exercise 4) and 5), under the same assumptions find the drift of the Brownian motion W_t under the probability measure $P_T^{h,A}$ defined on $\mathcal{F}_T^W$ as

$$P_T^{h,A}(d\omega) = \frac{h(W_T)\exp(A_T)}{E_P(h(W_T)\exp(A_T))} P_T(d\omega) = Z_T(\omega) P_T(d\omega)$$

with h and A as in exercise 4 and 5 .