UNIVERSITY OF HELSINKI

Department of Mathematics and Statistics
Stochastic Analysis, spring 2008
Solutions to Problem Sheet 10 (Apr 23, 2008)

1. *Of course, we need to assume that* $f(X,Y) \in L^1(\mathbf{P})$. Let us denote $h(x) := \mathbf{E}[f(x,Y)]$ and let $A \in \sigma(X)$, that is $A = X^{-1}(B)$ for some $B \in \mathcal{B}(\mathbb{R})$. Now, by independence and Fubini's theorem,

$$\begin{aligned}
\mathbf{E}[\mathbf{1}_A f(X,Y)] &= \int_{\mathbb{R}^2} \mathbf{1}_B(x) f(x,y) \mathbf{P}(X \in \mathrm{d}x, Y \in \mathrm{d}y) \\
&= \int_{\mathbb{R}^2} \mathbf{1}_B(x) f(x,y) \mathbf{P}(X \in \mathrm{d}x)\mathbf{P}(Y \in \mathrm{d}y) \\
&= \int_{\mathbb{R}} \mathbf{1}_B(x) \left(\int_{\mathbb{R}} f(x,y) \mathbf{P}(Y \in \mathrm{d}y) \right) \mathbf{P}(X \in \mathrm{d}x) \\
&= \int_{\mathbb{R}} \mathbf{1}_B(x) h(x) \mathbf{P}(X \in \mathrm{d}x) \\
&= \mathbf{E}[\mathbf{1}_A h(X)].
\end{aligned}$$

Hence, the assertion follows from the definition of the conditional expectation.

2. i) When X_t is multiplied by the integrating factor e^{-t}, Itô's formula yields

$$\begin{aligned}
e^{-t} X_t &= X_0 - \int_0^t e^{-s} X_s b_s \mathrm{d}s + \int_0^t e^{-s} \mathrm{d}X_s \\
&= X_0 - \int_0^t e^{-s} X_s b_s \mathrm{d}s + \int_0^t e^{-s} X_s b_s \mathrm{d}s + \int_0^t e^{-s} f_s \mathrm{d}V_s + \int_0^t e^{-s} g_s \mathrm{d}W_s,
\end{aligned}$$

from which we obtain

$$X_t = e^t X_0 + \int_0^t e^{t-s} f_s \mathrm{d}V_s + \int_0^t e^{t-s} g_s \mathrm{d}W_s. \tag{1}$$

ii) First, we define

$$B_t := \int_0^t \frac{f_s}{\sqrt{f_s^2 + g_s^2}} \mathrm{d}V_s + \int_0^t \frac{g_s}{\sqrt{f_s^2 + g_s^2}} \mathrm{d}W_s.$$

For its bracket, we have (cf. Exercise 9.2.ii) $\langle B \rangle_t = t$. Since B is, by construction, a local martingale, Lévy's characterization theorem implies that it is a Brownian motion. Moreover, by associativity of stochastic integrals,

$$\begin{aligned}
X_t &= X_0 + \int_0^t X_s b_s \mathrm{d}s + \int_0^t f_s \frac{\sqrt{f_s^2 + g_s^2}}{\sqrt{f_s^2 + g_s^2}} \mathrm{d}V_s + \int_0^t g_s \frac{\sqrt{f_s^2 + g_s^2}}{\sqrt{f_s^2 + g_s^2}} \mathrm{d}W_s \\
&= X_0 + \int_0^t X_s b_s \mathrm{d}s + \int_0^t \sqrt{f_s^2 + g_s^2} \mathrm{d} \left(\int_0^\cdot \frac{f_u}{\sqrt{f_u^2 + g_u^2}} \mathrm{d}V_u \right)_s \\
&\quad + \int_0^t \sqrt{f_s^2 + g_s^2} \mathrm{d} \left(\int_0^\cdot \frac{g_u}{\sqrt{f_u^2 + g_u^2}} \mathrm{d}W_u \right)_s \\
&= X_0 + \int_0^t X_s b_s \mathrm{d}s + \int_0^t \sqrt{f_s^2 + g_s^2} \mathrm{d}B_s.
\end{aligned}$$

iii) First we show that (X, W) is a two-dimensional Gaussian process. To this end, let $d \in \mathbb{N} \setminus \{0\}$, $t_1, \ldots, t_d \in \mathbb{R}_+$, $\alpha_1, \ldots, \alpha_d \in \mathbb{R}$, $\beta_1, \ldots, \beta_d \in \mathbb{R}$, and $T := \max\{t_i : i = 1, \ldots, d\}$.

Now, we can write

$$\begin{aligned} Z := \sum_{i=1}^d (\alpha_i X_{t_i} + \beta_i W_{t_i}) &= \sum_{i=1}^d \alpha_i e^{t_i} X_0 + \sum_{i=1}^d \alpha_i \int_0^{t_i} e^{t_i - s} f_s \mathrm{d}V_s + \sum_{i=1}^d \alpha_i \int_0^{t_i} e^{t_i - s} g_s \mathrm{d}W_s \\ &\quad + \sum_{i=1}^d \beta_i \int_0^{t_i} \mathrm{d}W_s \\ &= \left(\sum_{i=1}^d \alpha_i e^{t_i} \right) X_0 + \int_0^T \sum_{i=1}^d \mathbf{1}_{(0,t_i]}(s) \alpha_i e^{t_i - s} f_s \mathrm{d}V_s \\ &\quad + \int_0^T \sum_{i=1}^d \mathbf{1}_{(0,t_i]}(s) (\alpha_i e^{t_i - s} g_s + \beta_i) \mathrm{d}W_s. \end{aligned}$$

By Exercise 3.5, the stochastic integrals on the right hand side are Gaussian. Since X_0, $(V_t)_{t\geq 0}$, and $(W_t)_{t\geq 0}$ are independent, Z is a sum of three independent Gaussian random variables, and is thus Gaussian. This shows that (X, W) is a Gaussian process.

For (X, Y), we can write

$$\begin{aligned} Z' := \sum_{i=1}^d (\alpha_i X_{t_i} + \beta_i Y_{t_i}) &= \sum_{i=1}^d \alpha_i X_{t_i} + \sum_{i=1}^d \beta_i \int_0^{t_i} X_s h_s \mathrm{d}s + \sum_{k=1}^d \beta_i W_{t_i} \\ &= \sum_{i=1}^d \alpha_i X_{t_i} + \int_0^T X_s \underbrace{\sum_{i=1}^d \mathbf{1}_{(0,t_i]}(s) \beta_i h_s}_{=:a(s)} \mathrm{d}s + \sum_{k=1}^d \beta_i W_{t_i}, \end{aligned}$$

where obviously $a \in L^2([0,T])$. Now, there exists a sequence $(a_n) \subset C^\infty([0,T])$ such that $\|a - a_n\|_{L^2([0,T])} \to 0$. By the Cauchy–Schwarz inequality,

$$\begin{aligned} \left(\int_0^T X_s a(s) \mathrm{d}s - \int_0^T X_s a_n(s) \mathrm{d}s \right)^2 &= \left(\int_0^T X_s \big(a(s) - a_n(s)\big) \mathrm{d}s \right)^2 \\ &\leq \int_0^T X_s^2 \mathrm{d}s \times \|a - a_n\|_{L^2([0,T])}^2, \end{aligned}$$

and consequently,

$$\left\| \int_0^T X_s a(s) \mathrm{d}s - \int_0^T X_s a_n(s) \mathrm{d}s \right\|_{L^2(\Omega)} \leq \mathbf{E}\left[\int_0^T X_s^2 \mathrm{d}s \right] \|a - a_n\|_{L^2([0,T])}^2 \xrightarrow[n\to\infty]{} 0,$$

where, by Fubini's theorem and (1),

$$\mathbf{E}\left[\int_0^T X_s^2 \mathrm{d}s \right] = \int_0^T \mathbf{E}[X_s^2] \mathrm{d}s \leq T e^{2T} \left(\mathbf{E}[X_0^2] + (\|f\|_{L^2([0,T])}^2 + \|g\|_{L^2([0,T])}^2) \right) < \infty.$$

Next, we show that $\int_0^T X_s a_n(s) \mathrm{d}s$ can be defined as an L^2-limit of Riemann sums (by continuity of the integrand, we already know that it exists as an *almost sure* limit of Riemann sums).[1] To this end, let $\pi_m := \{0 = s_0^{(m)} < s_1^{(m)} < \cdots < s_m^{(m)} = T\}$ be such that $\lim_{m\to\infty} \mathrm{mesh}(\pi_m) = 0$, and define

$$I_m^{(n)} := \sum_{j=1}^m X_{s_j} a_n(s_j)(s_j - s_{j-1}).$$

Now, we have

$$\mathbf{E}[I_m^{(n)} I_{m'}^{(n)}] = \sum_{j=1}^m \sum_{j'=1}^{m'} \mathbf{E}[X_{s_j} X_{s_{j'}}] a_n(s_j) a_n(s_{j'})(s_j - s_{j-1})(s_{j'} - s_{j'-1}),$$

[1]For this paragraph, we follow the lecture notes G. Lindgren: *Lectures on Stationary Stochastic Processes*, available from `http://www.maths.lth.se/matstat/staff/georg/Publications/lecture2006.pdf`.

which is, for sufficiently large m and m', arbitrarily close to the planar integral

$$\int_0^T \int_0^T \mathbf{E}[X_s X_{s'}] a_n(s) a_n(s') \mathrm{d}s \mathrm{d}s' =: J^{(n)}$$

(this holds because the covariance function $(s, s') \mapsto \mathbf{E}[X_s X_s']$ is continuous — which one can check by computing $\mathbf{E}[X_s X_{s'}]$). Now, let $\varepsilon > 0$ and $\overline{m}$ be such that $|\mathbf{E}[I_m^{(n)} I_{m'}^{(n)}] - J^{(n)}| < \varepsilon/4$ for all $m, m' > \overline{m}$. Hence, for any $m, m' > \overline{m}$ we have

$$\begin{aligned}
0 \leq \mathbf{E}[(I_m^{(n)} - I_{m'}^{(n)})^2] &= |\mathbf{E}[I_m^{(n)} I_m^{(n)}] - \mathbf{E}[I_m^{(n)} I_{m'}^{(n)}] - \mathbf{E}[I_{m'}^{(n)} I_m^{(n)}] + \mathbf{E}[I_{m'}^{(n)} I_{m'}^{(n)}]| \\
&\leq |\mathbf{E}[I_m^{(n)} I_m^{(n)}] - \mathbf{E}[I_m^{(n)} I_{m'}^{(n)}]| + |\mathbf{E}[I_{m'}^{(n)} I_m^{(n)}] - \mathbf{E}[I_{m'}^{(n)} I_{m'}^{(n)}]| \\
&\leq |\mathbf{E}[I_m^{(n)} I_m^{(n)}] - J^{(n)}| + |\mathbf{E}[I_m^{(n)} I_{m'}^{(n)}]| - J^{(n)}| \\
&\quad + |\mathbf{E}[I_{m'}^{(n)} I_m^{(n)}] - J^{(n)}| + |\mathbf{E}[I_{m'}^{(n)} I_{m'}^{(n)}] - J^{(n)}| < \varepsilon.
\end{aligned}$$

This shows that $(I_m^{(n)})_{m=1}^\infty$ is a Cauchy sequence in $L^2(\Omega)$, and hence

$$I_m^{(n)} \xrightarrow[m \to \infty]{L^2} \int_0^T X_s a_n(s) \mathrm{d}s.$$

Now we can write

$$Z' = L^2\text{-} \lim_{n \to \infty} \left\{ L^2\text{-} \lim_{m \to \infty} \left(\sum_{i=1}^d \alpha_i X_{t_i} + I_m^{(n)} + \sum_{k=1}^d \beta_i W_{t_i} \right) \right\},$$

where

$$\sum_{i=1}^d \alpha_i X_{t_i} + I_m^{(n)} + \sum_{k=1}^d \beta_i W_{t_i} = \sum_{i=1}^d \alpha_i X_{t_i} + \sum_{j=1}^m X_{s_j} a_n(s_j)(s_j - s_{j-1}) + \sum_{k=1}^d \beta_i W_{t_i},$$

which is a Gaussian random variable, since (X, W) is, as we showed, a Gaussian process. Now, recall that the set of Gaussian random variables on Ω is a closed linear subspace of $L^2(\Omega)$ (see the solution to the Exercise 5.3). Hence Z' is Gaussian, and thus (X, Y) is a Gaussian process.

3. i) For simplicity, let us take $k = 2$ (for larger values of k, finding an Itô representation of $h(W_T)$ is a more involved task — the point of this exercise is how to apply *Girsanov's theorem*). Using Itô's formula, we obtain

$$W_T^2 = T + \int_0^T 2 W_t \mathrm{d}W_t.$$

Hence,

$$\frac{\mathrm{d}\mathbf{P}_T^h}{\mathrm{d}\mathbf{P}_T} = \frac{W_T^2}{\mathbf{E}[W_T^2]} = 1 + \int_0^T \frac{2W_t}{T} \mathrm{d}W_t,$$

from which we obtain

$$D_t := \mathbf{E}\left[\frac{\mathrm{d}\mathbf{P}_T^h}{\mathrm{d}\mathbf{P}_T} \middle| \mathcal{F}_t^W \right] = 1 + \int_0^t \frac{2W_s}{T} \mathrm{d}W_s.$$

ii) By Girsanov's theorem,

$$\begin{aligned}
W_t^h &:= W_t - \int_0^t D_s^{-1} \mathrm{d} \underbrace{\langle W, D \rangle_s}_{= \int_0^s \frac{2W_u \mathrm{d}u}{T}} \\
&= W_t - \int_0^t \frac{2W_s \mathrm{d}s}{T(1 + \int_0^s \frac{2W_u}{T} \mathrm{d}W_u)} = W_t - \int_0^t \frac{W_s \mathrm{d}s}{\frac{T}{2} + \int_0^s W_u \mathrm{d}W_u}
\end{aligned}$$

defines a local $\mathbf{P}_T^h$-martingale $(W_t^h)_{t\in[0,T]}$. Hence, the drift of (W_t) under the measure $\mathbf{P}_T^h$ is given by the process

$$\int_0^\cdot \frac{W_s \mathrm{d}s}{\frac{T}{2} + \int_0^s W_u \mathrm{d}W_u}.$$

Remark 2. By Lévy's characterization theorem, W^h is a Brownian motion under the measure $\mathbf{P}_T^h$.

4. i) We will obtain an expression for $\mathbf{E}[\exp(A_T)]$ in as a 'by-product' in part ii. Hence, we will omit the computation at this point.

   ii) Integration by parts yields

$$\begin{aligned} A_t = \int_0^t W_s \mathrm{d} \underbrace{\left(\int_0^\cdot f(u)\mathrm{d}u\right)_s}_{=:C_s} &= C_t W_t - \int_0^t C_s \mathrm{d}W_s - \underbrace{\langle C, W\rangle_t}_{=0} \\ &= \int_0^t (C_t - C_s)\mathrm{d}W_s \\ &= \int_0^t \int_s^t f(u)\mathrm{d}u\mathrm{d}W_s. \end{aligned}$$

Hence,

$$\begin{aligned} \exp(A_T) &= \exp\left(\int_0^T \int_s^T f(u)\mathrm{d}u\mathrm{d}W_s\right) \\ &= \exp\left(\frac{1}{2}\int_0^T \left(\int_s^T f(u)\mathrm{d}u\right)^2 \mathrm{d}s\right) \mathcal{E}\left(\int_0^\cdot \int_s^T f(u)\mathrm{d}u\mathrm{d}W_s\right)_T. \end{aligned}$$

Now, since the stochastic exponential is a (uniformly integrable) martingale starting from unity, by Novikov's criterion (see Revuz–Yor, Proposition VIII.1.15),

$$\begin{aligned} \mathbf{E}[\exp(A_T)] &= \exp\left(\frac{1}{2}\int_0^T \left(\int_s^T f(u)\mathrm{d}u\right)^2 \mathrm{d}s\right) \underbrace{\mathbf{E}\left[\mathcal{E}\left(\int_0^\cdot \int_s^T f(u)\mathrm{d}u\mathrm{d}W_s\right)_T\right]}_{=1} \\ &= \exp\left(\frac{1}{2}\int_0^T \left(\int_s^T f(u)\mathrm{d}u\right)^2 \mathrm{d}s\right). \end{aligned}$$

Consequently,

$$\tilde{D}_t := \mathbf{E}\left[\frac{\mathrm{d}\mathbf{P}_T^A}{\mathrm{d}\mathbf{P}_T}\middle|\mathcal{F}_t^W\right] = \mathbf{E}\left[Z_T\middle|\mathcal{F}_t^W\right] = \mathcal{E}\left(\int_0^\cdot \underbrace{\int_s^T f(u)\mathrm{d}u}_{=:F_s}\mathrm{d}W_s\right)_t.$$

By Girsanov's theorem,

$$W_t^A = W_t - \int_0^t \tilde{D}_s^{-1}\mathrm{d}\langle W, \tilde{D}\rangle_s$$

defines a local $\mathbf{P}_T^A$-martingale $(W_t^A)_{t\in[0,T]}$. Now, recall that

$$\tilde{D}_t = \mathcal{E}\left(\int_0^\cdot F_s \mathrm{d}W_s\right)_t = 1 + \int_0^t \mathcal{E}\left(\int_0^\cdot F_u \mathrm{d}W_u\right)_s F_s \mathrm{d}W_s = 1 + \int_0^t \tilde{D}_s F_s \mathrm{d}W_s$$

(see Exercise 6.1). Hence,

$$\langle W, \tilde{D}\rangle_t = \int_0^t \tilde{D}_s F_s \mathrm{d}s,$$

and moreover, the drift is

$$\int_0^\cdot \tilde{D}_s^{-1}\mathrm{d}\langle W, \tilde{D}\rangle_s = \int_0^\cdot \tilde{D}_s^{-1}\mathrm{d}\left(\int_0^\cdot \tilde{D}_u F_u \mathrm{d}u\right)_s = \int_0^\cdot F_s \mathrm{d}s.$$

Remark 3. Analogously to Exercise 3, by Lévy's characterization theorem, W^A is a Brownian motion under the measure $\mathbf{P}_T^A$.

5. First, write
$$\frac{\mathrm{d}\mathbf{P}_T^{h,A}}{\mathrm{d}\mathbf{P}_T} = \frac{h(W_T)\exp(A_T)}{\mathbf{E}[h(W_T)\exp(A_T)]} = \frac{h(W_T)\mathbf{E}[\exp(A_T)]}{\mathbf{E}[h(W_T)\exp(A_T)]}\frac{\exp(A_T)}{\mathbf{E}[\exp(A_T)]} = \frac{\mathrm{d}\mathbf{P}_T^{h,A}}{\mathrm{d}\mathbf{P}_T^A}\frac{\mathrm{d}\mathbf{P}_T^A}{\mathrm{d}\mathbf{P}_T}.$$
In the previous exercise, we showed that
$$W_t = W_t^A + \alpha_t, \tag{4}$$
where $(W_t^A)_{t\in[0,T]}$ is a Brownian motion under $\mathbf{P}_T^A$, and $\alpha_t = \int_0^t F_s \mathrm{d}s$. In particular, we note that the drift α is deterministic. Now, it suffices to find the drift of W^A under the measure $\mathbf{P}_T^{h,A}$. To this end, we should find a martingale representation for $\mathrm{d}\mathbf{P}_T^{h,A}/\mathrm{d}\mathbf{P}_T^A$. However, this is straightforward, since
$$\begin{aligned} h(W_T) = (W_T^A + \alpha_T)^2 = (W_T^A)^2 + 2W_T^A\alpha_T + \alpha_T^2 \\ &= \int_0^T 2W_s^A \mathrm{d}W_s^A + T + \int_0^T 2\alpha_T \mathrm{d}W_s^A + \alpha_T^2 \\ &= \int_0^T 2(W_s^A + \alpha_T)\mathrm{d}W_s^A + T + \alpha_T^2. \end{aligned}$$
Hence,
$$\bar{D}_t := \mathbf{E}_{\mathbf{P}_T^A}\left[\frac{\mathrm{d}\mathbf{P}_T^{h,A}}{\mathrm{d}\mathbf{P}_T^A}\middle|\mathcal{F}_t^W\right] = \mathbf{E}_{\mathbf{P}_T^A}\left[\frac{\mathrm{d}\mathbf{P}_T^{h,A}}{\mathrm{d}\mathbf{P}_T^A}\middle|\mathcal{F}_t^{W^A}\right] = 1 + \int_0^t \frac{2(W_s^A + \alpha_T)}{T + \alpha_T^2}\mathrm{d}W_s^A$$
(since α_t is deterministic, $\mathcal{F}_t^W = \mathcal{F}_t^{W^A}$). By Girsanov's theorem
$$\begin{aligned} W_t^{A,h} &= W_t^A - \int_0^t \tilde{D}_s^{-1}\mathrm{d}\langle \tilde{D}, W^A\rangle_s \\ &= W_t^A - \int_0^t \frac{(W_s^A + \alpha_T)\mathrm{d}s}{\frac{T+\alpha_T^2}{2} + \int_0^s (W_u^A + \alpha_T)\mathrm{d}W_u^A} \end{aligned}$$
defines a Brownian motion $(W_t^{A,h})_{t\in[0,T]}$ under $\mathbf{P}_T^{A,h}$. Together with (4), this shows that the drift of W under $\mathbf{P}_T^{A,h}$ is given by
$$\alpha + \int_0^{\cdot} \frac{(W_s^A + \alpha_T)\mathrm{d}s}{\frac{T+\alpha_T^2}{2} + \int_0^s (W_u^A + \alpha_T)\mathrm{d}W_u^A}.$$