UNIVERSITY OF HELSINKI

Department of Mathematics and Statistics
Stochastic Analysis, Spring 2008
Solutions to Problem Sheet 1 (Jan 29, 2007)

1. Let us first recall some preliminaries. Observe, that the set

$$C := \Big\{ x \in \mathbb{R}^T : (x(t_1), \dots, x(t_d)) \in B \Big\},$$

where $B \in \mathcal{B}(\mathbb{R}^d)$, can be represented as

$$C = \Big\{ x \in \mathbb{R}^T : e_F(x) \in B \Big\} = e_F^{-1}(B), \tag{1}$$

where $e_F : \mathbb{R}^T \to \mathbb{R}^F$ is the *evaluation map* and $F := \{t_1, \dots, t_d\}$. Of course, here we have identified $\mathbb{R}^F$ and $\mathbb{R}^d$, but this does not cause any problems, since $(\mathbb{R}^F, \mathcal{B}(\mathbb{R}^F))$ and $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ are Borel isomorphic — that is, the natural identification map from $\mathbb{R}^F$ to $\mathbb{R}^d$ and its inverse are Borel measurable.[1] Finally, let π^I_J denote *projection* from $\mathbb{R}^I$ onto $\mathbb{R}^J$, where $\emptyset \neq J \subset I$. If S is such that $F \subset S \subset T$, the diagram

$$\begin{tikzcd}
\mathbb{R}^T \arrow[r, "\pi^T_S"] \arrow[dr, "e_F=\pi^T_F"'] & \mathbb{R}^S \arrow[d, "\pi^S_F"] \\
& \mathbb{R}^F
\end{tikzcd} \tag{2}$$

commutes. If S is finite, then $\pi^T_S = e_S$ is measurable, by the definition of the product σ-algebra. Moreover, π^S_F is obviously continuous, and hence measurable.

Let us begin by checking that $\mathcal{C}$ is closed under complementation. This is straightforward, since by (1) we have

$$C^{\mathrm{c}} = \left(e_F^{-1}(B)\right)^{\mathrm{c}} = e_F^{-1}(B^{\mathrm{c}}) \in \mathcal{C},$$

as $B^{\mathrm{c}} \in \mathcal{B}(\mathbb{R}^F)$. Next we check that $\mathcal{C}$ is closed under finite unions. By induction, it suffices to show this for unions of two sets. To this end, define $F' := \{t'_1, \dots, t'_{d'}\}$ and

$$C' := \Big\{ x \in \mathbb{R}^T : \big(x(t'_1), \dots, x(t'_{d'})\big) \in B' \Big\},$$

where $B' \in \mathcal{B}(\mathbb{R}^{F'})$. Now by (2), we can write $C = e^{-1}_{F\cup F'}(\tilde{B})$ and $C' = e^{-1}_{F\cup F'}(\tilde{B}')$, where $\tilde{B} := (\pi^{F\cup F'}_F)^{-1}(B)$ and $\tilde{B}' := (\pi^{F\cup F'}_{F'})^{-1}(B')$, respectively. Since $\tilde{B}, \tilde{B}' \in \mathcal{B}(\mathbb{R}^{F\cup F'})$, we have

$$C \cup C' = e^{-1}_{F\cup F'}(\tilde{B}) \cup e^{-1}_{F\cup F'}(\tilde{B}') = e^{-1}_{F\cup F'}(\tilde{B} \cup \tilde{B}') \in \mathcal{C},$$

as $\tilde{B} \cup \tilde{B}' \in \mathcal{B}(\mathbb{R}^{F\cup F'})$.

2. By (1), we see that for evaluation maps, preimages of Borel sets are cylinder sets and, conversely, that cylinder sets are preimages of Borel sets. Since the generators of $\mathcal{B}(\mathbb{R})^{\otimes T}$ and $\sigma(\mathcal{C})$ thus coincide, we have $\mathcal{B}(\mathbb{R})^{\otimes T} = \sigma(\mathcal{C})$.

3. By induction, it suffices to show additivity for unions of two sets. Let C and C' be as in the first problem, however, with the additional assumption that $C \cap C' = \emptyset$. We define

$$P(C \cup C') := \mu_{F\cup F'}(\tilde{B} \cup \tilde{B}') = \mu_{F\cup F'}\left((\pi^{F\cup F'}_F)^{-1}(B) \cup (\pi^{F\cup F'}_{F'})^{-1}(B') \right). \tag{3}$$

---

[1] This follows from the fact that $\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R})^{\otimes d}$.

Since $C \cap C' = \emptyset$, we must have $\tilde{B} \cap \tilde{B}' = \emptyset$. Hence, by consistency,

$$\begin{aligned}\mu_{F\cup F'}(\tilde{B} \cup \tilde{B}') = \mu_{F\cup F'}(\tilde{B}) + \mu_{F\cup F'}(\tilde{B}') &= \mu_{F\cup F'}\left((\pi_F^{F\cup F'})^{-1}(B)\right) + \mu_{F\cup F'}\left((\pi_{F'}^{F\cup F'})^{-1}(B')\right)\\ &= \mu_F(B) + \mu_{F'}(B') = P(C) + P(C').\end{aligned}$$

Note that consistency of the finite dimensional distributions guarantees that P is well-defined on $\mathcal{C}$, and in particular, that (3) is a valid way to define P for unions. To see this, suppose that $C := e_F^{-1}(B) = e_{\hat{F}}^{-1}(\hat{B})$, where $\hat{B} \in \mathcal{B}(\mathbb{R}^{\hat{F}})$ and $\hat{F} \subset T$ is finite. By (2),

$$(\pi_{F\cup\hat{F}}^T)^{-1}\left((\pi_F^{F\cup\hat{F}})^{-1}(B)\right) = e_F^{-1}(B) = e_{\hat{F}}^{-1}(\hat{B}) = (\pi_{F\cup\hat{F}}^T)^{-1}\left((\pi_{\hat{F}}^{F\cup\hat{F}})^{-1}(\hat{B})\right),$$

from which we have

$$(\pi_F^{F\cup\hat{F}})^{-1}(B) = (\pi_{\hat{F}}^{F\cup\hat{F}})^{-1}(\hat{B}).$$

Hence, by consistency,

$$\mu_F(B) = \mu_{F\cup\hat{F}}\left((\pi_F^{F\cup\hat{F}})^{-1}(B)\right) = \mu_{F\cup\hat{F}}\left((\pi_{\hat{F}}^{F\cup\hat{F}})^{-1}(\hat{B})\right) = \mu_{\hat{F}}(\hat{B}).$$

4. i) We begin by recalling an auxiliary result from basic point-set topology. The (rather simple) proof of this lemma can be found e.g. from DUGUNDJI: *Topology* (Theorem III.3.1) or VÄISÄLÄ: *Topologia II* (in Finnish).

Lemma 4. *Let $\mathcal{A}$ be a cover of a set X. Then there is a topology $\mathcal{T}$ on X such that*
(a) *$\mathcal{A}$ is a subbase for $\mathcal{T}$,*
(b) *$\mathcal{T}$ is the coarsest topology on X, for which $\mathcal{A} \subset \mathcal{T}$.*

Since family $\mathcal{O} := \{e_t^{-1}(U) : U \subset \mathbb{R} \text{ is open}, t \in T\}$ is a cover of $\mathbb{R}^T$, and since any topology with respect to which the evaluation maps are continuous must contain $\mathcal{O}$, by Lemma 4, $\mathcal{O}$ is a subbase for the product topology $\mathcal{T}_p$. Now, observe that

$$A := \left\{x \in \mathbb{R}^T : x_{t_1} \in U_1, \dots, x_{t_d} \in U_d\right\}$$

for some indices $t_1, \dots, t_d \in T$ and open sets $U_1, \dots, U_d \subset \mathbb{R}$ can be written as

$$A = \bigcap_{i=1}^{d} e_{t_i}^{-1}(U_i).$$

Hence $A \in \mathcal{T}_p$ and the topology $\mathcal{T}_p'$, generated by sets with the same structure as A, is not finer than the product topology $\mathcal{T}_p$. On the other hand, $\mathcal{O}$ is contained in $\mathcal{T}_p'$, so $\mathcal{T}_p$ cannot be finer than $\mathcal{T}_p'$. Hence $\mathcal{T}_p = \mathcal{T}_p'$.

ii) Let us denote

$$\mathcal{G} := \left\{A \in \mathcal{B}(\mathbb{R}^T) : A = (\pi_S^T)^{-1}(B), B \in \mathcal{B}(\mathbb{R}^S), S \subset T \text{ is countable}\right\}.$$

Clearly, $\mathcal{C} \subset \mathcal{G}$. Next we show that $\mathcal{G}$ is a σ-algebra. Obviously, it is nonempty. For any $A = (\pi_S^T)^{-1}(B)$, where $B \in \mathcal{B}(\mathbb{R}^S)$, we have

$$A^c = \left((\pi_S^T)^{-1}(B)\right)^c = (\pi_S^T)^{-1}\underbrace{(B^c)}_{\in\mathcal{B}(\mathbb{R}^S)} \in \mathcal{G}.$$

For every $i \in \mathbb{N}$, pick a set $B_i \in \mathcal{B}(\mathbb{R}^{S_i})$, where $S_i \subset T$ is countable. Moreover, define $A_i := (\pi_{S_i}^T)^{-1}(B_i)$. Next, we want to show that $\pi_{S_j}^{\cup_i S_i}$ is continuous (and hence measurable). Note that $\cup_i S_i$ is countable. It suffices to check that the preimage of a typical element U in the subbase of the product topology on $\mathbb{R}^{S_j}$ is open in $\mathbb{R}^{\cup S_i}$. So, consider $U := (\pi_{\{t\}}^{S_j})^{-1}(V)$, where $t \in T$ and $V \subset \mathbb{R}$ is open. But now

$$(\pi_{S_j}^{\cup_i S_i})^{-1}(U) = (\pi_{S_j}^{\cup_i S_i})^{-1}\left((\pi_{\{t\}}^{S_j})^{-1}(V)\right) = (\pi_{\{t\}}^{\cup_i S_i})^{-1}(V),$$

which is open, since it belongs to a subbase for the product topology on $\mathbb{R}^{\cup_i S_i}$. Returning to σ-algebras, we have now that

$$\begin{aligned}\bigcup_{j\in\mathbb{N}} A_j &= \bigcup_{j\in\mathbb{N}} (\pi^T_{\cup S_i})^{-1}\left((\pi^{\cup_i S_i}_{S_j})^{-1}(B_j)\right)\\ &= (\pi^T_{\cup S_i})^{-1}\Bigg(\underbrace{\bigcup_{j\in\mathbb{N}}(\pi^{\cup_i S_i}_{S_j})^{-1}(B_j)}_{\in\mathcal{B}(\mathbb{R}^{\cup_i S_i})}\Bigg)\in\mathcal{G},\end{aligned}$$

by the continuity of the projections. We have now established that $\sigma(\mathcal{C})\subset\mathcal{G}$. It remains to show the converse inclusion, $\mathcal{G}\subset\sigma(\mathcal{C})$, or just that for any countable $S\subset T$, we have $\sigma(\pi^T_S)\subset\sigma(\mathcal{C})$. But since $\mathcal{B}(\mathbb{R}^S)$ is generated by preimages of finite-dimensional Borel set under projections, the inclusion is clear (why?).

iii) Suppose that $C([0,1],\mathbb{R})\in\sigma(\mathcal{C})$. This implies that there is a countable set $S\subset[0,1]$ and $B\in\mathcal{B}(\mathbb{R}^S)$ such that $C([0,1],\mathbb{R}) = (\pi^{[0,1]}_S)^{-1}(B)$. Now pick $f\in C([0,1],\mathbb{R})$ and $y\in[0,1]\setminus S$. Define

$$\tilde{f}(x) := \begin{cases} f(x)+1, & x=y,\\ f(x), & x\neq y.\end{cases}$$

Clearly, $\tilde{f}:[0,1]\to\mathbb{R}$ cannot be continuous, yet $\pi^{[0,1]}_S\circ f = \pi^{[0,1]}_S\circ\tilde{f}$, which implies that $\tilde{f}\in(\pi^{[0,1]}_S)^{-1}(B) = C([0,1],\mathbb{R})$ — a contradiction.

Finally, we will show that $C([0,1],\mathbb{R})\in\mathcal{B}(\mathbb{R}^{[0,1]})$ (the following argument was hinted at in the book 'DUDLEY: *Real Analysis and Probability*'). One checks easily that

$$C([0,1],\mathbb{R}) = \bigcap_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\underbrace{\left\{f\in\mathbb{R}^{[0,1]} : |x-y|\le\frac{1}{m}\Rightarrow|f(x)-f(y)|\le\frac{1}{n}\right\}}_{:=C_{n,m}}.$$

We just need to show that $C_{n,m}$ is a closed subset of $\mathbb{R}^{[0,1]}$ for all $m,n\ge1$. So pick $f\notin C_{n,m}$ (obviously such a function exists). By definition, there are $x',y'\in[0,1]$ such that $|x'-y'|\le 1/m$, but $|f(x')-f(y')|>1/n$. Without loss of generality, assume $f(x')>f(y')$. Clearly there exist $a,b\in\mathbb{R}$, such that $f(x')>a>b>f(y')$ and $a-b>1/n$. Now, define $A := e^{-1}_{y'}((-\infty,b))\cap e^{-1}_{x'}((a,\infty))$. We see that A is a open subset of $\mathbb{R}^{[0,1]}$, $f\in A$, and $A\cap C_{n,m}=\emptyset$ (since $|g(x')-g(y')|>a-b>1/n$ for all $g\in A$). Hence, the complement of $C_{n,m}$ is open, and by definition, $C_{n,m}$ is closed.