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Consider the one period market with two assets $(S^0, S^1) > 0$ at time $t = 1$, and initial prices (π^0, π^1) at time $t = 0$ on a probability space $(\Omega, \mathcal{F}, P)$.

We assume that $\pi^0 = \pi^1 = 1$, and that $S^0 = (1 + r)$ is a bank account with deterministic interest rate r, while $S^1(\omega) = \exp(Y(\omega))$ with Y gaussian random variable with 0 mean and variance 1.

We have seen that this model is arbitrage free and incomplete.

Assume now that european call options $(S^1(\omega) - k)^+$ are traded in the market at prices $c^{call}(k)$ for all possible strike prices $k > 0$.

1): What is the condition on the european call option prices in order to obtain an extended market free of arbitrage ?

The condition would be that there is an equivalent risk neutral probability measure $Q \sim P$ such that simultaenously we have

$$E_Q(S^1) = (1+r)\frac{\pi_1}{\pi_0} = (1+r) \text{ and } E_Q((S^1 - k)^+) = (1+r)\frac{c(k)}{\pi_0} = (1+r)c(k)$$

Now, since under P the r.v. S^1 has strictly positive probability density $p(x)$ with $p(x) = 0$ for $x \le 0$ and $p(x) > 0$ for $x > 0$, the same holds under the measure Q, so that denoting by $q(x)$ the probability density of S^1 under Q, we must have

$$\begin{aligned}
&q(x) = 0 \text{ for } x \le 0, \quad q(x) > 0 \text{ for } x > 0 \\
&\int_0^\infty q(x)dx = 1 \\
&\int_k^\infty (x-k)q(x)dx = (1+r)c(k)\ , \quad k > 0 \\
&\int_0^\infty xq(x)dx = (1+r)
\end{aligned}$$

Now the price system $(\pi_0, \pi_1, c(k) : k > 0)$ is free of arbitrage if and only if there is at least one measurable function $q(x)$ satisfying these conditions.

How to check this in practice could be a good master thesis topic.

2) Compute the arbitrage free price of the contingent claim

$$X(\omega) = \{(S^1(\omega))^2 - k_0\}^+$$

for a fixed k_0 in the extended market, where at time $t=0$ we can trade stocks the instruments S^0, S^1 at prices π_0, π_1 and all possible european call option $(S^1(\omega)-v)^+$ at prices $c^{call}(v)$ for all $v \geq 0$.

Note that $f(x)=(x^2-k)^+$ is convex.
Note that for $x>\sqrt{k}$

$$(x^2-k)^+ = 2\int_{\sqrt{k}}^{x} u du = 2\int_{\sqrt{k}}^{x}\int_0^u dv du = 2\int_0^x \int_{\sqrt{k}\vee v}^{x} du dv =$$
$$= 2\int_0^x (x-\sqrt{k}\vee v) dv = 2\int_0^{+\infty} (x-\sqrt{k}\vee v)^+ dv$$

where $u \vee v = \max(u,v)$, while $(x^2-k)^+=0$ for $x<\sqrt{k}$, and the formula also holds.

This means that the fair price of the contingent claim $X(\omega)=(S^1(\omega)^2-k)^+$ is

$$c(X) = 2\int_0^\infty c^{call}(\sqrt{k}\vee v)dv =$$
$$= 2c^{call}(\sqrt{k})\sqrt{k} + 2\int_{\sqrt{k}}^\infty c^{call}(v)dv$$

where $c^{call}(v)$ was the market price of the european call option $(S^1(\omega)-v)$.

This means that to hedge the contract $X(\omega)$ we buy at time 0 $2\sqrt{k}$ european call option with strike price $\sqrt{k}$ and $2dv$ call options with strike price v for all $v>\sqrt{k}$. Note that this replicating strategy is not necessarly unique. By assigning the price $c(X)$ to the contract X, the market remains arbitrage free if the europian call-option market was arbitrage free.

3) Prove Proposition 3.21 ii) in Tommi Sottinen lecture notes: If $X(\omega)\geq 0$ and $E(X^2)<\infty$, $k>0$ is the strike price of the europan put option.

If the constant $0<\alpha<1$ satifies

$$E(X) = E(\alpha\{X+(k-X)^+\}) = \mu$$

then

$$\mathrm{Var}(X) \geq \mathrm{Var}(\alpha\{X+(k-X)^+\})$$

Solution:

Note that by the put call parity for european options,

$$X - k = (X-k)^+ - (k-X)^+$$

so that

$$X + (k-X)^+ = k + (X-k)^+ = X \vee k = \max(X,k)$$

so we have that $E(X) = \alpha E(X \vee k)$, with $0 < \alpha < 1$.

Denote $U(\omega) = \alpha X(\omega) \vee k$,

$$a_X(y) = \int_0^y x\ F_X(dx), \quad a_U(y) = \alpha \int_0^y (x \vee k)\ F_X(dx),$$

Now since by construction $a_X(+\infty) = a_U(+\infty) = E(X) = \mu$, it follows that $a_X(y) \leq a_U(y)$ for all y since we note that for $y < \alpha k$ necessarly $a_U(y) \geq a_X(y)$, since $a_U(0) = a_X(0) = 0$, and $a'_U(y) \geq a'_X(y)$, for $y < \alpha k$. If we had $a_U(y^*) > a_X(y^*)$, necessarly we would have $a_U(y) > a_X(y)$ for all $y > a^*$, since it would follow that $a'_U(y) \geq a'_X(y)$ for $y > a^*$, but this is not possible because of the boundary condition $a_X(+\infty) = a_U(+\infty) = E(X)$.

If we take the right-continuous inverses, we have $a_U^{-1}(a) \leq a_X^{-1}(a)$ for all a.

Then we get

$$E(X^2) = \int_0^\infty x^2 F_X(dx) = \int_0^\infty x a_X(dx) = \int_0^\mu a_X^{-1}(a)da \geq \int_0^\mu a_U^{-1}(a)da = \int_0^\infty x a_U(dx) = \alpha \int_0^\infty x(x \vee k) F_X(dx) = \alpha E(X(X \vee k))$$

and also

$$E(X(X \vee k)) = \int_0^\infty (x \vee k) x F_X(dx) = \int_0^\mu (a_X^{-1}(a) \vee k)da$$
$$\geq \int_0^\mu (a_U^{-1}(a) \vee k)da = \alpha \int_0^\mu (x \vee k)(x \vee k) F_x(dx) = \alpha E(X^2 \vee k^2)$$

Therefore we get

$$E(X^2) \geq \alpha E(X(X \vee k)) \geq \alpha^2 E(X^2 \vee k^2) = E(U^2)$$

which gives $\mathrm{Var}(X) \geq \mathrm{Var}(U)$ since $E(X) = E(U) = \mu$.

4) Assume that $(X_s : s \in \mathbb{N})$ is a sequence of independent random variables under the probability measure P with $E(|X_s|) < \infty$ and $E(X_s) = a_s$.

Show that

$$M_t(\omega) = \sum_{s=1}^{t} (X_s(\omega) - a_s)$$

is a martingale in the filtration $\{\mathcal{F}_t : t \in \mathbb{N}\}$ with $\mathcal{F}_t = \sigma(X_1, \dots X_t)$.

Now M_t is $\mathcal{F}_t$ measurable for each t since it is sum of $(X_s(\omega) - a_s) \in \mathcal{F}_s \subseteq \mathcal{F}_t$, $s = 1, \dots, t$.

M_t is integrable, since by the triangle inequality

$$E(|M_t|) \leq \sum_{i=1}^{t} \{E(|X_s|) + |a_s|\} < \infty$$

since X_s are integrable, and the martingale property follows since

$$M_t = M_{t-1} + (X_t - a_t)$$

and taking conditional expectation

$$E(M_t|\mathcal{F}_{t-1}) = M_{t-1} + E(X_t|\mathcal{F}_{t-1}) - a_t = M_{t-1} + E(X_t) - a_t = M_{t-1}$$

by using the properties of the conditional expectation and the fact that under P X_t is independent from the σ-algebra $\mathcal{F}_{t-1} = \sigma(X_1, \dots, X_{t-1})$.

5) Under the assumption of exercise 4)

Show that

$$N_t(\omega) = \prod_{s=1}^{t} \left(\frac{X_s(\omega)}{a_s} \right)$$

is a martingale in the filtration $\{\mathcal{F}_t : t \in \mathbb{N}\}$

(we must assume that $a_s \neq 0$).

Now N_t is $\mathcal{F}_t$ measurable for each t since for $s = 1, \dots, t$ $X_s(\omega)a_s^{-1}$ is $\mathcal{F}_s$-measurable and therefore $\mathcal{F}_t$-measurable and the product of measurable random variables is measurable.

We have to show that N_t is integrable when the r.v. $X_s, s = 1, \dots, t$ are integrable.

In general $X, Y \in L^1(P)$ does not imply that the product $(XY) \in L^1(P)$.

However, in this case using independence integrability follows since

$$E(|N_t|) = \frac{E(|X_1|)E(|X_2|)\dots E(|X_t|)}{|a_1 a_2 \dots a_t|}$$

which is finite when $a_s \neq 0$ and $E(|X_s|) < \infty$ for $s = 1, \dots, t$.

To check the martingale property, we use the representation

$$N_t = N_{t-1} X_t / a_t$$

taking now conditional expectation, using the properties of the conditional expectation and the fact that X_t is independent from $\mathcal{F}_{t-1}$,

$$E(N_t | \mathcal{F}_{t-1}) = N_{t-1} E(X_t | \mathcal{F}_{t-1}) / a_t = N_{t-1} E(X_t) / a_t = N_{t-1} \ .$$