SOME BASIC FACTS FROM MARTINGALE THEORY

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1. Conditional Expectation and Martingales

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 1. *Conditional expectation: Let X be a random variable, (which is $\mathcal{F}$-measurable) and a sub σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, $E_P(X|\mathcal{G})$ is a $\mathcal{G}$-measurable random variable such that for all $B \in \mathcal{G}$*

$$E_P(\mathbf{1}_B X) = E_P(\mathbf{1}_B E_P(X|\mathcal{G}))$$

Properties: i) $E_P(E_P(X|\mathcal{G})) = E_P(X)$,
ii) if Y is $\mathcal{G}$-measurable $E_P(XY|\mathcal{G}) = Y E_P(X|\mathcal{G})$.
iii) if $Y \perp\!\!\!\perp \mathcal{G}$, $E_P(Y|\mathcal{G}) = E_P(Y)$.

iv) If $E_P(X^2) < \infty$, the random variable $E_P(X|\mathcal{G})$ is the orthogonal projection of the r.v. X to the subspace $L^2(\Omega, \mathcal{G}, P) \subset L^2(\Omega, \mathcal{F}, P)$:

$$E((X - E_P(X|\mathcal{G}))^2) = \min_{Y \in L^2(\Omega, \mathcal{G}, P)} E((X-Y)^2) \ .$$

v) the conditional expectaion is linear:
$E_P(aX + bY|\mathcal{G})(\omega) = aE_P(X|\mathcal{G})(\omega) + bE_P(Y|\mathcal{G})(\omega)$
vi) The conditional expectaion is linear is non-negative, if $X(\omega) \geq 0$ P a.s. , then $E(X|\mathcal{G})(\omega) \geq 0$ P a.s.
Let Q a probability measure which dominates P ($P \ll Q$) on a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, which means that $Q(A) = 0 \implies P(A) = 0$ for all $A \in \mathcal{G}$. The Radon-Nikodym derivative of P w.r.t Q is a $\mathcal{G}$-measurable random variable

$$Z^{\mathcal{G}}(\omega) = Z^{\mathcal{G}}(P,Q)(\omega) = \frac{dP|\mathcal{G}}{dQ|\mathcal{G}}(\omega) \geq 0$$

This means that $P(d\omega) = Z(P,Q)(\omega)Q(d\omega)$ on $\mathcal{G}$, and if X is a $\mathcal{G}$-measurable random variable we change the measure to represent the expectation w.r.t. P as an expectation w.r.t. Q:

$$E_P(X) = E_Q(XZ(P,Q))$$

We have that $0 \leq Z^{\mathcal{G}}(P,Q) \in L^1(\Omega, \mathcal{G}, Q)$, ja $E_Q(Z(P,Q)) = 1$.
In statistics $Z(P,Q)$ is called likelihood ratio.
Note that if $\mathcal{A} \subseteq \mathcal{G}$ and $P \ll Q$ on $\mathcal{G}$, then trivially $P \ll Q$ on $\mathcal{A}$, and

$$Z^{\mathcal{A}}(P,Q) = E_Q(Z^{\mathcal{G}}(P,Q)|\mathcal{A}).$$

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This is the Q-martingale property for nested σ-algebrae.

We have also a formula to change the measure in the conditional expectation. For $P \ll Q$, $\mathcal{G} \subseteq \mathcal{F}$, and X is $\mathcal{F}$-measurable, *Bayes formula* holds:

$$E_P(X|\mathcal{G}) = \frac{E_Q(XZ(P,Q)|\mathcal{G})}{E_Q(Z(P,Q)|\mathcal{G})}$$

Sometimes it is also called abstract Bayes formula. The proof is not difficult, for $B \in \mathcal{G}$, denoting $Z = Z^{\mathcal{F}}(P,Q)$,

$$E_P(X\mathbf{1}_B) = E_Q(ZX\mathbf{1}_B) = E_Q(E_Q(ZX\mathbf{1}_B|\mathcal{G})) = E_Q(E_Q(ZX|\mathcal{G})\mathbf{1_B})$$
$$= E_Q\left(\frac{E_Q(Z|\mathcal{G})}{E_Q(Z|\mathcal{G})}E_Q(ZX|\mathcal{G})\mathbf{1}_B\right) = E_Q\big(Z\frac{E_Q(ZX|\mathcal{G})}{E_Q(Z|\mathcal{G})}\mathbf{1}_B\big) = E_P\big(\frac{E_Q(ZX|\mathcal{G})}{E_Q(Z|\mathcal{G})}\mathbf{1}_B\big)$$

and the result follows from the definition of conditional expectation.

Example 1. *As an exercise we show that the elementary Bayes formula used in statistics follows as a special case:*
Let (X,Y) a random vector with values in $\mathbb{R}^2$, with

$$P(X \in dx, Y \in dy) = \pi(x)p(y|x)dxdy$$

We work directly on the canonical space $\Omega = \mathbb{R}^2$. On the σ-algebra $\mathcal{F} = \sigma(X,Y)$, we take as reference measure a dominating product measure, for example $Q(dx,dy) = \pi(x)dxdy$ (although Q is not a probability measure, Bayes formula works also in this case).
Clearly $P \ll Q$ and $Z(P,Q) = \frac{dP}{dQ}(x,y) = p(y|x)$.
When we condition to the sub-σ-algebra $\mathcal{G} = \sigma(Y)$, our (abstract) Bayes formula says that for any bounded measurable function $f(x)$,

$$E_P(f(X)|\sigma(Y))(\omega) = \frac{E_Q(f(X)Z(P,Q)|\sigma(Y))(\omega)}{E_Q(Z(P,Q)|\sigma(Y))(\omega)} = \frac{\int_{\mathbb{R}} f(x)\pi(x)p(Y(\omega)|x)dx}{\int_{\mathbb{R}} \pi(x)p(Y(\omega)|x)dx}$$

which is the elementary Bayes formula as we use it in statistics.

We introduce now a filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$, which is an increasing sequence of σ-algebrae shich that, for all $s \leq t$, $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$.
(here it does not matter whether the time is discrete or continuous, we can always imbed discrete time in continuous time by taking $\mathcal{F}_t = \mathcal{F}_{\lfloor t \rfloor}$).

Definition 2. *A process M_t is a $(P,\mathbb{F})$-martingale if M_t is $\mathcal{F}_t$ measurable, $M_t \in L^1(P)$, and for $s \leq t$*

$$E_P(M_t|\mathcal{F}_s) = M_s \,.$$

When

$$E_P(M_t|\mathcal{F}_s) \leq M_s \quad , \; s \leq t$$

we say that (M_t) is a $(P,\{\mathcal{F}_t\})$-supermartingale, and if

$$E_P(M_t|\mathcal{F}_s) \geq M_s \quad , \; s \leq t$$

(M_t) is a $(P,\{\mathcal{F}_t\})$-submartingale.

Given all the past, the conditional expectation of a future value of a martingale is the current value.
Note that the martingale property depends on the measure P and on the filtration $\{\mathcal{F}_t\}$.
Given two measures P and Q defined on $(\Omega, \mathcal{F})$ we consider at each time t the restriction of the measures to the current information σ-algebra $\mathcal{F}_t$, $P_t = P|_{\mathcal{F}_t}$, $Q_t = Q|_{\mathcal{F}_t}$.
If $P_t \ll Q_t$ on $\mathcal{F}_t$, we define

$$Z_t(P, Q) = \frac{dP_t}{dQ_t}.$$

From the definition it follows that $Z_t \in L^1(Q, \mathcal{F}_t)$ and $Z_t(\omega) \geq 0$.
We show that Z_t is a $(Q, \mathbb{F})$ martingale: for $s \leq t$ if $B \in \mathcal{F}_s$ also $B \in \mathcal{F}_t$ and we have

$$P(B) = E_P(\mathbf{1}_B) = E_Q(Z_s \mathbf{1}_B) = E_Q(Z_t \mathbf{1}_B)$$

which means that $Z_s = E_Q(Z_t | \mathcal{F}_s)$.

Example 2. *On a probability space $(\Omega, \mathcal{F})$ we have a sequence of (real valued) random variables $(X_1, X_2, \dots, X_n, \dots)$, and two probability measures P and Q such that (X_i) are independent and identically distributed under both P and Q. We assume that $P(X_1 \in dx) = f(x) Q(X_1 \in dx)$. Let $\mathcal{F}_t = \sigma(X_1, \dots, X_t)$, $t \in \mathbb{N}$. It follows that*

$$Z_t(P, Q) = \prod_{s \in \mathbb{N}: s \leq t} f_s(X_i) \ .$$

Exercise 1. *Check that $Z(P, Q)$ is a $(Q, \{\mathcal{F}_t\})$-martingale.*

Definition 3. *We say that a process (X_t) is adapted if $X_t \in \mathcal{F}_t$ for all t, and in the discrete-time situation it is predictable if $X_t \in \mathcal{F}_{t-1}$ for all t.*

Theorem 1.1. *(discrete-time Doob-Meyer decomposition).*
If (X_t) is adapted to the filtration $\{\mathcal{F}_t\}$, and $E(|X_t|) < \infty$ for all $t = 0, 1, \dots, T$ then there is an unique decomposition

$$X_t = X_0 + A_t + M_t$$

where A_t is $\{\mathcal{F}_t\}$-predictable and M_t is a $\{\mathcal{F}_t\}$-martingale with $A_0 = 0$ and $M_0 = 0$.
If (X_t) is a supermartingale (respectively submartingale) the process A_t is non-increasing, (respectively non-decrasing submartingale).

Proof

$$\Delta X_t = (\Delta X_t - E_P(\Delta X_t | \mathcal{F}_{t-1})) + E_P(\Delta X_t | \mathcal{F}_{t-1}) = \Delta M_t + \Delta A_t$$

where

$$A_t = \sum_{s=1}^{t} E_P(\Delta X_t | \mathcal{F}_{t-1}), \quad M_t = \sum_{s=1}^{t} \big(\Delta X_t - E_P(\Delta X_t | \mathcal{F}_{t-1}) \big)$$

If another Doob decomposition of X existed, $X_t - X_0 = \tilde{A}_t + \tilde{M}_t$ we would have $(M_t - \tilde{M}_t) = (A_t - \tilde{A}_t)$ which means that $(M_t - \tilde{M}_t)$ is a predictable martingale, which is necessarily the constant zero.

Definition 4. *If (Y_t) and (X_t) are sequences we define the stochastic integral of Y with respect to X as the sequence*

$$(Y \cdot X)_t = \sum_{s=1}^{t} Y_s \Delta X_s$$

which is called martingale transform or discrete stochastic integral

Theorem 1.2. *Assume that (Y_t) $\{\mathcal{F}_t\}$-predictable process and (M_t) is a $(P, \{\mathcal{F}_t\})$-martingale. If Y_t is a bounded random variable for all t, or alternatively both Y_t and M_t are square integrable r.v., it follows that $E(|Y_t \Delta M_t|) < \infty$. Under such assumptions, the stochastic integral $(Y \cdot M)_t$ is a martingale.*

Proof: Exercise.

2. Square integrable martingales and predictable bracket

A $(P, \{\mathcal{F}_t\})$-martingale (M_t) is square integrable when $E(M_t^2) < \infty$ for all t.
If M_t, N_t are square integrable martingales then by using Cauchy-Schwartz inequality

$$E(|M_t N_t|) \leq \sqrt{E(M_t^2)} \sqrt{E(N_t^2)} < \infty$$

so that the product $(M_t N_t)$ is in L^1 and it makes sense to consider its Doob-Meyer decomposition:
We have

$$M_t N_t - M_{t-1} N_{t-1} = M_{t-1} \Delta N_t + N_{t-1} \Delta M_t + \Delta M_t \Delta N_t =$$
$$M_{t-1} \Delta N_t + N_{t-1} \Delta M_t + \big(\Delta M_t \Delta N_t - E_P(\Delta M_t \Delta N_t | \mathcal{F}_{t-1}) \big) + E_P(\Delta M_t \Delta N_t | \mathcal{F}_{t-1})$$

We introduce the predictable process

$$\langle M, N \rangle_t := \sum_{s=1}^{t} E_P(\Delta M_s \Delta N_s | \mathcal{F}_{s-1})$$

We obtain the Doob-Meyer decomposition

$$M_t N_t = M_0 N_0 + \langle M, N \rangle_t + m_t$$

where dm_t the sum the martingale increments

$$dm_t = M_{t-1} \Delta N_t + N_{t-1} \Delta M_t + \big(\Delta M_t \Delta N_t - E_P(\Delta M_t \Delta N_t | \mathcal{F}_{t-1}) \big)$$

where the integrability conditions in the definition of martingale follow from Cauchy-Schwartz inequality since we have assumed M and N are square-integrable.
We denote also

$$[M, N]_t := \sum_{s=1}^{t} \Delta M_s \Delta N_s$$

it follows that the process $([M,N]_t - \langle M,N\rangle_t)$ is a $(P,\{\mathcal{F}_t\})$-martingale.

$[M,N]_t$ is called quadratic covariation or square-bracket process, while $\langle M,N\rangle_t$ is called predictable covariation, or predictable-bracket process.
Since $E((\Delta M_t)_P|\mathcal{F}_{t-1}) \geq 0$, the process $([M,M]_t)$ is a submartingale and therefore $(\langle M,M\rangle_t)$ is non-decreasing. The notations $[M]_t := [M,M]_t$ and $\langle M\rangle_t := \langle M,M\rangle_t$ are also used.
Note $[M,N]_t$ does not depend on the measure P, but the predictable bracket $\langle M,N\rangle_t$ does !

Definition 5. *Two square integrable martingales $(M_t),(N_t)$ are orthogonal if the product (M_tN_t) is a martingale. Equivalent conditions are*
i) $[M,N]_t$ is a martingale,
ii) $\langle M,N\rangle_t = 0$, which means $E_P(\Delta M_t\Delta N_t|\mathcal{F}_{t-1})(\omega) = 0$ P a.s.

3. Orthogonal projections in the space of square integrable martingales

Let M and N two square integrable martingales,
We write

$$N_t = N_0 + (H\cdot M)_t + N_t^{\perp} = N_0 + \sum_{s=1}^{t} H_s\Delta M_s + N_t^{\perp}$$

where (H_t) is the predictable process

$$H_t = \mathbf{1}\big(\Delta\langle M,M\rangle_t > 0\,\big)\,\frac{\Delta\langle M,N\rangle_t}{\Delta\langle M,M\rangle_t} = \mathbf{1}\big(E_P((\Delta M_t)^2|\mathcal{F}_{t-1}) > 0\,\big)\,\frac{E_P(\Delta M_t\Delta N_t|\mathcal{F}_{t-1})}{E_P((\Delta M_t)^2|\mathcal{F}_{t-1})}$$

and $N_t^{\perp}$ is a P-martingale orthogonal to M_t.

4. Martingale property and change of measure

Theorem 4.1. *Let $Q \ll P$ and let*

$$Z_t(\omega) = Z_t(Q,P) = \frac{dQ_t}{dP_t}(\omega)$$

Then M_t is a $(Q,\{\mathcal{F}_t\})$-martingale if and only if the product (M_tZ_t) is a $(P,\{\mathcal{F}_t\})$-martingale.

Proof for $s \leq t$, let $A \in \mathcal{F}_s$.

$$E_Q(1_A(M_t - M_s)) = E_P(1_AZ_t(M_t - M_s)) = E_P(1_A(Z_tM_t - Z_sM_s))$$

where we use the properties of the conditional expectation. By definition of conditional expectation it means that

$$E_Q(M_t|\mathcal{F}_s) = M_s \text{ if and only if } E_P(Z_tM_t|\mathcal{F}_s) = Z_sM_s$$

5. Doob decomposition and change of measure

Suppose that M is a square integrable $(P, \mathcal{F}_t)$ martingale with $M_0 = 0$ and $\Delta M_t > -1$.

$$Z_t = \mathcal{E}(M)_t := \prod_{s=1}^{t} (1 + \Delta M_t) = \left(1 + \sum_{s=1}^{t} Z_{s-1} \Delta M_s\right) > 0$$

and we define on each $\mathcal{F}_t$ consistently a probability measure

$$Q_t(d\omega) = Z_t(\omega) P_t(d\omega)$$

where $Q_t(\Omega) = E_P(Z_t) = Z_0 = 1$.
Assume that N is another square integrable P martingale. By projectiing on M obtaining the orthogonal martingale decomposition

$$N_t = N_0 + (H \cdot M)_t + N_t^{\perp}$$

What happens to the martingale property of N and M under the new measure ?

Proposition 5.1. *(Girsanov theorem in discrete time) The Doob decomposition of N under Q is given by*

$$N_t = N_0 + \big(H \cdot \langle M, M \rangle\big)_t + \big(H \cdot (M - \langle M, M \rangle)\big)_t + N_t^{\perp}$$

where $(M - \langle M, M \rangle)_t$ is a Q-martingale and $N^{\perp}$ is a martingale under both P and Q, and $(H \cdot \langle M, M \rangle)_t$ is a predictable process.

Proof

$$E_Q(\Delta M_t | \mathcal{F}_{t-1}) = E_P\big(\Delta M_t \frac{Z_t}{Z_{t-1}} | \mathcal{F}_{t-1}\big) = E_P\big(\Delta M_t \big(1 + \frac{\Delta Z_t}{Z_{t-1}}\big) | \mathcal{F}_{t-1}\big) =$$
$$E_P(\Delta M_t | \mathcal{F}_{t-1}) + E_P((\Delta M_t)^2 | \mathcal{F}_{t-1}) = \ 0 \ + \ \Delta \langle M, M \rangle_t$$

which means that $(M_t - \langle M, M \rangle_t)$ is a Q-martingale.
On the other hand

$$E_Q(\Delta N_t^{\perp} | \mathcal{F}_{t-1}) = E_P(\Delta N_t^{\perp} \Delta M_t | \mathcal{F}_{t-1}) = \Delta \langle N^{\perp}, M \rangle_t = 0$$

since $N^{\perp}$ and M are orthogonal martingales.

6. Martingale predictable representation property

Let M be a P-martingale w.r.t. to a discrete time filtration $\{\mathcal{F}_t : t \in \mathbb{N}\}$. We say that M has the martingale representation property in the filtration $\mathbb{F} = \{\mathcal{F}_t\}$, if any other bounded $(P, \mathbb{F})$-martingale (X_t) can be represented as a constant plus a martingale transform w.r.t. M

$$X_t = X_0 + (Y \cdot M)_t = X_0 + \sum_{s=1}^{t} Y_s \Delta M_s$$

where (Y_t) is $\mathbb{F}$-predictable, that is Y_t is $\mathcal{F}_{t-1}$-measurable for all t. Note that this notation covers also the case of d-dimensional martingales. In such case (Y_s) is a d-dimensional predictable process, and

$$\sum_{s=1}^{t} Y_s \Delta M_s = \sum_{s=1}^{t} \sum_{i=1}^{d} Y_s^{(i)} \Delta M_s(i)$$

Lemma 6.1. *Let (M_t) be a $(P, \mathbb{F})$-martingale.*
(M_t) has the predictable representation property in the $(\mathbb{F})$-filtration if and only if
the only bounded $(P, \mathbb{F})$-martingales (N_t) such that the product $(M_t N_t)$ is a $(P, \mathbb{F})$-martingale are constant.

Proof Assume that the PRP holds for M. Then every bounded martingale N has the form $N_t = (H \cdot M)_t$. If N is such that $(N_t M_t)$ is a martingale, necessarily

$$\Delta(M_t N_t) = M_{t-1}\Delta N_t + N_{t-1}\Delta M_t + \Delta M_t \Delta N_t =$$
$$(M_{t-1}H_t + N_{t-1})\Delta M_t + H_t(\Delta M_t)^2$$

This gives a contradiction, since

$$0 = E(\Delta(M_t N_t)|\mathcal{F}_{t-1}) = H_t E((\Delta M_t)^2|\mathcal{F}_{t-1}) \neq 0$$

with positive probability unless either $\Delta M_t = 0$ or $H_t = 0$. This implies that N_t is constant. The same argument gives the opposite implication.

Theorem 6.1. *In the discrete time setting, M has the martingale representation property in the filtration $\mathbb{F}$ if and only if there are no other martingale measures $Q \sim P$ with bounded density for (M_t), that is if $Q \sim P$, $Z(\omega) = \frac{dP}{dQ}(\omega)$ is essentially bounded and (M_t) is a also a $(Q, \mathbb{F})$-martingale, necessarly $Q = P$.*

Proof For simplicity we set $\mathcal{F}_0 = \{\Omega, \emptyset\}$. Assume that $Q \sim P$. We know that $Z_t = Z_t(Q, P)$ is a $(P, \mathbb{F})$-martingale.
By the predictable representation property,

$$\Delta Z_t = Z_{t-1} H_t \Delta M_t$$

where H_t is $\mathcal{F}_{t-1}$-measurable.
We show that M is not a martingale under Q, unless $H_t = 0$.

$$E_Q(\Delta M_t|\mathcal{F}_{t-1}) = E_P\big(\Delta M_t \frac{Z_t}{Z_{t-1}}|\mathcal{F}_{t-1}\big) = E_P\big(\Delta M_t\big(1 + \frac{\Delta Z_t}{Z_{t-1}}\big)|\mathcal{F}_{t-1}\big) =$$
$$E_P\big(\Delta M_t\big(1 + H_t \Delta M_t\big)|\mathcal{F}_{t-1}\big) = E_P(\Delta M_t|\mathcal{F}_{t-1}) + E_P(H_t(\Delta M_t)^2|\mathcal{F}_{t-1}) =$$
$$0 + H_t E_P((\Delta M_t)^2|\mathcal{F}_{t-1}) \neq 0$$

unless $H_t = 0$ P-a.s. for all t. This means that $Z_t = 1$ for all t and $Q = P$.

Viceversa, suppose that the representation property does not hold for M in the filtration $\mathbb{F}$.
This means that there is some other bounded $(P, \mathbb{F})$-martingale N such that the product $(M_t N_t)$ is a martingale. We can take N satisfying $N_0 = 0$ and $|N_t| \leq 1$. It is a fact from martingale theory that a bounded martingale (N_t) has almost surely a limit.
Define the measure on $\mathcal{F}_t$

$$dQ_t = \left(1 + \frac{N_t}{2}\right) dP_t = Z_t(\omega) dP_t$$

Note that (Z_t) is a P-martingale with $0 < \frac{1}{2} \leq Z_t(\omega) \leq 3/2$ and $Z_0 = 1$, so that Q_t is a probabilty measure equivalent to P_t on $\mathcal{F}_t$.

We have that

$$M_tZ_t = M_t + \frac{(N_tM_t)}{2}$$

is a P-martingale since (M_t) and (N_tM_t) are P-martingales. This means we have constructed another measure $Q_t \sim P_t$, with $Q_t \neq P_t$ such that (M_t) is a Q-martingale.

Example 3. *Consider a sequence of i.i.d. standard normal random variables (ξ_t) on the probability space $(\Omega, \mathcal{F}, P)$. with the filtration of σ algebrae $\mathcal{F}_t = \sigma(\xi_s : 1 \leq s \leq t)$.*
Define $M_t = \sum_{s=1}^{t} \xi_s$. M_t is a P-martingale, since it has independent increments and centered. M_t is also square integrable, since the increments are gaussian. Note that $\mathcal{F}_t = \sigma(M_s : 1 \leq s \leq t)$.
Note that $\eta_t = (\xi_t^2 - 1)$ are also i.i.d. and centered, and $N_t = \sum_{s=1}^{t} \eta_s$ is also a P-martingale.
It follows that the product (N_tM_t) is a P-martingale,m since $E_P(\xi_t\eta_t) = E_P(\xi_t^3 - \xi_t) = 0$.
The filtration $\{\mathcal{F}_t\}$ generated by (M_t) contains the P-martingale (N_t) which is is orthogonal to (M_t). Neither M or N have the predictable representation property.
We show that there exist an equivalent martingale measure for M. Note that $\Delta N_t = (\xi_t^2 - 1) > -1$ P-almost surely.
Therefore

$$Z_t = \prod_{s=1}^{t}(1 + \Delta N_t) = 1 + \sum_{s=1}^{t} Z_{s-1}\Delta N_s > 0$$

defines an equivalent probability measure $dQ_t = Z_t dP_t$.
By Girsanov theorem, since (M_tN_t) is a P-martingale it follows that also (M_tZ_t) is a P-martingale. But this means that (M_t) is a Q-martingale. So $Q \sim P$ but $Q \neq P$ is another martingale measure for P.
In order to construct a bounded $(P, \{\mathcal{F}_t\})$- martingale we can take the i.i.d. sequence of centered and bounded random variables

$$\varepsilon_t := (\xi_t^2 \wedge 1) - E_P(\xi_t^2 \wedge 1) \in (-1, 1)$$

It follows that

$$E_P(\xi_t\varepsilon_t) = E_P(\xi_t(\xi_t^2 \wedge 1)) - E_P(\xi_t)E_P(\xi_t^2 \wedge 1) =$$
$$E_P(\xi_t \mathbf{1}(|\xi_t| > 1)) + E_P(\xi_t^3 \mathbf{1}(|\xi_t| \leq 1)) + 0 = 0$$

since the distribution ξ_t is symmetric around 0.
Therefore for any fixed T, the process stopped at T

$$X_t^T := \sum_{s=1}^{t\wedge T} \varepsilon_s$$

is a bounded P-martingale orthogonal to (M_t).

7. Application to hedging

Consider the finite probability space $(\Omega, \mathcal{F}, P)$ where $\Omega = \{0,1\}^T$, with $T < \infty$, and $\mathcal{F} = 2^\Omega$, the finite collection of all possible subset, and probability measure satisfies $P(\{\omega\}) > 0$ for all $\omega \in \Omega$.
An history is a vector $\omega = (\omega_1, \dots, \omega_T) \in \Omega$ and denote $\omega^t = (\omega_1, \dots, \omega_t)$ for $t \le T$.
Consider a market with a bank account B_t and a stock price S_t, $t = 0, 1, \dots, T$, adapted to the filtration $\mathbb{F}$ with $\mathcal{F}_t = \sigma(\omega_s, s \le t)$, $\mathcal{F}_0 = \{\Omega, \emptyset\}$
We assume that there are $\{\mathcal{F}_t\}$-**predictable** processes $U_t(\omega) > R_t(\omega) > D_t(\omega) > -1$. $B_0 > 0$ and $S_0 > 0$ are determistic values, and we let

$$B_t = B_0 \prod_{s=1}^{t} (1 + R_t),$$

$$S_t = S_0 \prod_{s=1}^{t} (1 + D_t + \omega_t (U_t - D_t))$$

Suppose that $G(\omega)$ is a $\mathcal{F}_t$-measurable contingent claim, and we want to find a self-financing hedging strategy (β_t, γ_t) satisfying

$$V_t = \beta_t B_t + \gamma_t S_t = \beta_{t+1} B_t + \gamma_{t+1} S_t \ .$$

We show first that there is an unique probability measure Q such that $Q \sim P$ and the discounted process $\bar{S}_t := (S_t / B_t)$ is a Q-martingale.
Once we have shown that Q is the unique martingale measure for $(\bar{S}_t)$ in the filtration $\mathbb{F}$, it follows that every $(Q, \mathbb{F})$ martingale (N_t) has the representation as

$$N_t = N_0 + \sum_{u=1}^{t} H_u \Delta \bar{S}_u$$

where (H_t) is a $\mathbb{F}$-predictable process. In particular we can take $N_t = E_Q(G|\mathcal{F}_t)$, and obtain when $t = T$

$$G(\omega) = E_Q(G|\mathcal{F}_T) = E_Q(G) + \sum_{t=1}^{T} \gamma_t \Delta \bar{S}_t$$

where (γ_t) is a $\mathbb{F}$-predictable process.
This gives the price and the hedging strategy for the contingent claim G.

Lets' first compute the martingale measure Q.

$$\Delta \bar{S}_t = \left(\frac{S_t}{B_t} - \frac{S_{t-1}}{B_{t-1}} \right) = \frac{S_{t-1}}{B_{t-1}} \left(\frac{(1 + D_t + (U_t - D_t)\omega_t)}{(1 + R_t)} - 1 \right) =$$
$$\frac{S_{t-1}}{B_{t-1}(1 + R_t)} \big((U_t - D_t)\omega_t - (D_t - R_t) \big)$$

Taking conditional expectation with respect to a measure Q, and imposing the martingale property

$$E_Q(\Delta \bar{S}_t | \mathcal{F}_{t-1}) = \frac{S_{t-1}}{B_{t-1}(1 + R_t)} \big((U_t - D_t) E_Q(\omega_t | \mathcal{F}_{t-1}) - (D_t - R_t) \big) = 0$$

which implies that Q is a martingale measure for $(\bar{S}_t)$ if and only if

$$q_t(\omega^{t-1}) := E_Q(\omega_t|\mathcal{F}_{t-1}) = \frac{(R_t - D_t)}{(U_t - D_t)},$$

where $q_t(\omega^{t-1}) \in (0,1)$ is a probability since we have assumed that $D_t < R_t < U_t$, P a.s, and it is uniquely determined. We define globally the unique risk-neutral measure Q as follows:

$$Q(\omega) = \prod_{t=1}^{T} q_t(\omega^{t-1})^{\omega_t}(1 - q_t(\omega^{t-1}))^{1-\omega_t}$$

and note that $Q(\{\omega\}) > 0$ for all $\omega \in \Omega$, therefore $Q \sim P$.
We define the basic Q-martingale

$$M_t = \sum_{s=1}^{t} \big(\omega_s - q_s(\omega^{(s-1)})\big)$$

We write

$$\Delta\bar{S}_t = \frac{S_{t-1}}{B_{t-1}(1+R_t)}(U_t - D_t)(\omega_t - q_t(\omega^{(t-1)})) = \frac{S_{t-1}}{B_{t-1}(1+R_t)}(U_t - D_t)\Delta M_t$$

and we can represent ΔM_t in terms of $\Delta\bar{S}_t$:

$$\Delta M_t = \frac{B_{t-1}(1+R_t)}{S_{t-1}(U_t - D_t)}\Delta\bar{S}_t$$

Next we show how to use the martingale representation to compute the hedging strategy for the contingent claim G.

Definition 6. *If $X(\omega)$ is a $\mathcal{F}_T$-measurable random variable, we define its discrete Malliavin derivative or stochastic gradient at time t w.r.t ω_t as*

$$\nabla_t X(\omega) := X(\omega_1, \dots, \omega_{t-1}, 1, \omega_{t+1}, \dots \omega_T) - X(\omega_1, \dots, \omega_{t-1}, 0, \omega_{t+1}, \dots \omega_T)\ ,$$

for $1 \le t \le T$.

Note that in general $\nabla_t X(\omega)$ is not $\mathcal{F}_t$ measurable unless the r.v. $X(\omega) = X(\omega^t)$ is $\mathcal{F}_t$-measurable. In such case $\nabla_t X(\omega)$ is also $\mathcal{F}_{t-1}$-measurable.
In particular the following quantities are $\mathcal{F}_{T-1}$-measurable.

$$\nabla_T G(\omega^{T-1}) = (G(\omega^{T-1},1) + G(\omega^{T-1},0)) \quad \text{and}$$
$$\nabla_T S_T(\omega^{T-1}) = (S_T(\omega^{T-1},1) + S_T(\omega^{T-1},0)) = S_{T-1}(U_T(\omega^{T-1}) - D_T(\omega^{T-1}))\ .$$
$$\nabla_T \bar{S}_T(\omega^{T-1}) = \frac{1}{B_T}\nabla_T \bar{S}_T(\omega^{T-1})$$

Note also that

$$\Delta\bar{S}_T = (\bar{S}_T - \bar{S}_{T-1}) = \frac{S_{T-1}}{B_T}(U_T - D_T)(\omega_T - q_T) = \nabla_T \bar{S}_T(\omega_T - q_T)$$

so that we can write

$$\Delta M_T = (\omega_T - q_T(\omega^{T-1})) = \frac{1}{\nabla_T \bar{S}_T}\Delta\bar{S}_T = \frac{B_T}{\nabla_T S_T}\Delta\bar{S}_T$$

We have

$$G(\omega) = G(\omega^{T-1}, \omega_T) = G(\omega^{T-1}, 0) + (G(\omega^{T-1}, 1) + G(\omega^{T-1}, 0))\omega_T =$$
$$G(\omega^{T-1}, 0) + \nabla_T G(\omega^{T-1})\omega_T =$$
$$G(\omega^{T-1}, 0) + \nabla_T G(\omega^{T-1})q_T + \nabla_T G(\omega^{T-1})(\omega_T - q_T) =$$
$$E_Q(G|\mathfrak{F}_{T-1}) + \nabla_T G\ \Delta M_T = E_Q(G|\mathfrak{F}_{T-1}) + \frac{\nabla_T G}{\nabla_T S_T} B_T \Delta \bar{S}_T$$
$$= E_Q(G|\mathfrak{F}_{T-1}) + \frac{\nabla_T G}{\nabla_T S_T}\Delta S_T - \frac{\nabla_T G}{\nabla_T S_T} R_T S_{T-1}$$
$$= E_Q(G|\mathfrak{F}_{T-1}) + \frac{\nabla_T G}{\nabla_T S_T}\Delta S_T - \frac{\nabla_T G}{\nabla_T S_T}\frac{S_{T-1}}{B_{T-1}}\Delta B_t$$

By investing at time $(T-1)$ the value

$$c_{T-1}(G) = \frac{E_Q(G|\mathfrak{F}_{T-1})}{1+R_T}$$

we replicate the contingent claim G as follows: we buy the amount of stocks

$$\gamma_T = \frac{\nabla_T G}{\nabla_T S_T}$$

at price $\gamma_T S_{T-1}$ (if $\gamma_T < 0$ we short-sell stocks) , if necessary by borrowing from the bank at the predictable interest rate R_T, and buy the amount of

$$\beta_T = \frac{1}{B_{T-1}}\Big(c_{T-1}(G) - \gamma_T S_{T-1}\Big)$$

bonds at price B_{T-1}, so that our capital is

$$V_{T-1} = c_{T-1}(G) = \beta_T B_{T-1} + \gamma_T S_{T-1}$$

At time $(T-1)$ the value of our portfolio is

$$V_{T-1} = \beta_T B_{T-1} + \gamma_T S_{T-1} = c_{T-1}(G)$$

while at time T the value of the portfolio becomes

$$V_T = \beta_T B_T + \gamma_T S_T = \beta_T B_{T-1}(1+R_T) + \gamma_T S_{T-1} + \gamma_T \Delta S_T$$
$$= E_Q(G|\mathfrak{F}_{T-1}) - \gamma_T S_{T-1}(1+R_T) + \gamma_T S_{T-1} + \gamma_T \Delta S_T$$
$$= E_Q(G|\mathfrak{F}_{T-1}) - \gamma_T S_{T-1} R_T + \gamma_T \Delta S_T =$$
$$E_Q(G|\mathfrak{F}_{T-1}) + \gamma_T(S_T - (1+R_T)S_{T-1}) = E_Q(G|\mathfrak{F}_{T-1}) + B_T \gamma_T \Delta \bar{S}_T = G(\omega)$$

Remark The martingale measure Q when it is unique gives a device to compute the price and hedging strategy. In fact the price hedging can be computed without using probability, once we have assumed that all histories $\omega \in \Omega$ have positive probability:
A direct way to compute the hedging without using martingales is to solve at time T the system of equations:

$$G(\omega^{T-1}, 0) = B_T \beta_T + \gamma_T S_{T-1}(1 + D_T)$$
$$G(\omega^{T-1}, 1) = B_T \beta_T + \gamma_T S_{T-1}(1 + U_T)$$

By substracting these two equations we get

$$\gamma_T = \frac{\nabla_T G(\omega^{T-1})}{S_{T-1}(U_T - D_T)}$$

and if the two equations with respective weights $(1-q_T(\omega^{T-1}))$ corresponding to $\omega_T=0$ and $q_T(\omega^{T-1})$ corresponding to $\omega_T=1$ we obtain

$$\begin{aligned}\beta_T &= \frac{1}{B_T}\big(E_Q(G|\mathcal{F}_{T-1})-\gamma_T E_Q(S_T|\mathcal{F}_{T-1})\big)\\ &= \frac{1}{B_T}E_Q(G|\mathcal{F}_{T-1})-\gamma_T\frac{S_{T-1}}{B_{T-1}}\end{aligned}$$

combining these toghether we get the price of the contingent claim at time $(T-1)$:

$$c_{T-1}(G)=\beta_T B_{T-1}+\gamma_T S_{T-1}=\frac{1}{1+R_T}E_Q(G|\mathcal{F}_{T-1})$$

The martingale method has the advantage that it gives a probabilistic interpretation to the price of the contingent claim, which can be computed directly as a Q-expectation.
The other reason is that the martingale method can be extended to the continuous-time setting.

The price and the hedging strategy in the whole time interval $t=1,\dots,T$, is then obtained by induction:
Let $c_t(G)$ be the price of the contract G at time $t\le T$. This is a $\mathcal{F}_t$-measurable contingent claim. This means that are able to hedge the contingent claim G expiring at time T if and only if at time t we own a portfolio of value $c_t(G)$. By repeating the martingale argument or by writing directly the system of equations we find the price of the contract at time $(t-1)$ $c_{t-1}(G)$ and the replicating portfolio $\beta_t(\omega^{t-1}),\gamma_t(\omega^{t-1})$.
The advantage the martingale method is that enables to compute directly price and replicating strategy at all times t by computing Q-expectations.
The predictable representation property of the Q-martingale M gives

Theorem 7.1. *Discrete Clarck-Ocone formula:*

$$\begin{aligned}E_Q(G|\mathcal{F}_t)(\omega) &= E_Q(G)+\sum_{s=1}^{t}\nabla_s E_Q(G(\omega)|\mathcal{F}_s)\big(\omega_s-q_s(\omega^{s-1})\big)\\ &= E_Q(G)+\sum_{u=1}^{t}\frac{\nabla_u E_Q(G(\omega)|\mathcal{F}_u)}{\nabla_u\bar{S}_u}\,\Delta\bar{S}_u\end{aligned}$$

where by definition $\nabla_t E_Q(G(\omega)|\mathcal{F}_t)$ is $\mathcal{F}_{t-1}$-measurable.

We set

$$\gamma_t=\frac{\nabla_t E_Q(G(\omega)|\mathcal{F}_t)}{\nabla_t S_t}$$

This gives

$$\begin{aligned}V_t &= E_Q(G|\mathcal{F}_t)=E_Q(G|\mathcal{F}_{t-1})+\gamma_t B_t\Delta\bar{S}_t\\ &= \frac{E_Q(G|\mathcal{F}_{t-1})}{1+R_t}+\gamma_t\Delta S_t+\left(\frac{E_Q(G|\mathcal{F}_{t-1})}{1+R_t}-\gamma_t S_{t-1}\right)\frac{1}{B_{t-1}}\Delta B_t\\ &= V_{t-1}+\gamma_t\Delta S_t+\beta_t\Delta B_t\end{aligned}$$

where

$$\beta_t = \left(\frac{E_Q(G|\mathcal{F}_{t-1})}{1+R_t} - \gamma_t S_{t-1} \right) \frac{1}{B_{t-1}}$$

This means that to obtain a portfolio with value $E_Q(G|\mathcal{F}_t)$ at time t, we need to invest

$$c_{t-1} := E_Q(G|\mathcal{F}_{t-1})/(1+R_t)$$

at time $(t-1)$. Equivalently, to have $E_Q(G\frac{B_t}{B_T}|\mathcal{F}_t)$ in our portfolio at time t we need to invest the amount

$$E_Q(G\frac{B_{t-1}}{B_T}|\mathcal{F}_{t-1}) \quad \text{at time } (t-1) \ .$$

Inductively , to have $G = E_Q(G|\mathcal{F}_T)$ at time T we have to invest at time $s \le T$ the amount

$$c_t(G) = E_Q(G\frac{B_t}{B_T}|\mathcal{F}_t)$$

at time t.
The hedging at time $(t-1)$ is given by

$$\gamma_t = \frac{\nabla_t E_Q(G(\omega)\frac{B_t}{B_T}|\mathcal{F}_t)}{\nabla_t S_t} = \frac{\nabla_t c_t(G)}{\nabla_t S_t},$$
$$\beta_t = \left(c_{t-1}(G) - \gamma_t S_{t-1} \right) \frac{1}{B_{t-1}}$$

giving

$$V_t = c_t(G) = c_0(G) + \sum_{u=1}^{t} \big(\gamma_u \Delta B_u + \beta_u \Delta B_u \big)$$
$$V_T = G = c_0(G) + \sum_{u=1}^{T} \big(\gamma_u \Delta B_u + \beta_u \Delta B_u \big)$$

When R_t is deterministic, we can take the discounting factors B_t/B_T outside the conditional expectation.

If (D_t, R_t, U_t) are all deterministic, then under the martingale measure Q the random variables ω_t is independent from the past. Then the computation of the hedging strategy may be simplified by using the following formula:

Corollary 7.1. *If (D_t, R_t, U_t) are deterministic at all $t \le T$, conditional expectation and gradient commute in Ito-Clarck formula*

$$\nabla_t E_Q(G|\mathcal{F}_t) = E_Q(\nabla_t G|\mathcal{F}_t) = E_Q(\nabla_t G|\mathcal{F}_{t-1}) \ ,$$

giving

$$E_Q(G|\mathcal{F}_t)(\omega) = E_Q(G) + \sum_{s=1}^{t} E_Q(\nabla_s G|\mathcal{F}_s)\big(\omega_s - q_s(\omega^{s-1})\big) \ .$$

Proof When $\omega = (\omega_1, \ldots, \omega_T)$ we denote $\omega^{t,T}$ the vector $(\omega_t, \ldots, \omega_T)$.
Using the independence of the r.v. (ω_t),

$$E_Q(\nabla_t G|\mathcal{F}_t)(\omega_t) = \sum_{\omega^{t+1,T}\in\{0,1\}^{T-t}} \big\{G(\omega^{t-1},1,\omega^{t+1,T}) - G(\omega^{t-1},0,\omega^{t+1,T})\big\}Q(\omega^{t+1,T})$$
$$= \nabla_t E_Q(G|\mathcal{F}_t)(\omega_t)$$

which is $\mathcal{F}_{t-1}$-measurable.

Example 4. *Assume that $R_t = r, U_t = u, D_t = d$ deterministic, with $-1 < d < r < u$. Then $q_t = q = (r-d)/(u-d)$ is constant. We have that*

$$S_t = S_0(1+u)^{N_t}(1+d)^{t-N_t}$$

where $N_t = \sum_{s=1}^{t} \omega_s$.
Then if $G(\omega) = \varphi(S_T)$ is a plain european option, we compute the price at time $t = 0$ using the distribution Binomial(q, T).

$$V_0 = c_0(G) = B_0 E_Q(\varphi(S_T)/B_T) =$$
$$(1+r)^{-T}\sum_{n=0}^{T}\binom{T}{n} q^n(1-q)^{T-n}\varphi\big(S_0(1+u)^n(1+d)^{T-n}\big) .$$

Similarly since the conditional distribution of $(N_T - N_t)$ given $\mathcal{F}_t$ is Binomial$(q, T - t)$, at time t the price of the replicating portfolio is

$$V_t = c_t(G) = B_t E_Q(\varphi(S_T)/B_T|\mathcal{F}_t) =$$
$$(1+r)^{t-T}\sum_{n=0}^{T-t}\binom{T-t}{n} q^n(1-q)^{T-t-n}\varphi\big(S_0(1+u)^{N_t+n}(1+d)^{T-N_t-n}\big) .$$

with this amount of money, we invest in γ_{t+1} stocks and invest the rest in the bank account, with

$$\gamma_{t+1} = \frac{\nabla_{t+1}c_{t+1}(G)}{\nabla_{t+1}S_{t+1}} = (1+r)^{t+1-T}\frac{E_Q(\nabla_{t+1}G|\mathcal{F}_t)}{S_t(u-d)} =$$
$$(1+r)^{t+1-T}\frac{1}{S_t(u-d)}\sum_{n=0}^{T-t-2}\Bigg\{\binom{T-t-2}{n}q^n(1-q)^{T-t-2-n}\times$$
$$\times\Big(\varphi\big(S_0(1+u)^{N_t+n+1}(1+d)^{T-N_t-n-2}\big) - \varphi\big(S_0(1+u)^{N_t+n}(1+d)^{T-N_t-n-1}\big)\Big)\Bigg\}$$