

**Mathematical Finance, Exercises -12 (10.12.08), Fall 2008.**

On a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$  we have two independent  $P$ -Brownian motions  $(W_t)$  and  $(B_t)$ .

Consider the following Black and Scholes market with two assets:

$$\begin{aligned}dS_t &= S_t(\mu dt + \sigma dW_t) \quad , S_0 > 0 \\dX_t &= X_t(b dt + a dB_t), \quad X_0 > 0\end{aligned}$$

where  $\mu, \sigma, a, b, \in \mathbb{R}$ ,  $(W_t)$  and  $(B_t)$  are independent a Brownian motions under the subjective probability measure  $P$ . Note that from independence it follows that  $\langle B, W \rangle_t = 0$ .

We have  $S_t, X_t > 0$  at all  $t$   $P$ -a.s. so we can choose each of them as numeraire. We want to see what happens with different choices of the numeraire. Note that there is not a riskless bank account which we could use.

**Exercise 1** Compute the stochastic differential equation satisfied by  $\tilde{X}_t = X_t/S_t$  and by  $\tilde{S}_t = S_t/X_t$

Hint: use either Ito formula or the integration by parts formula.

**Exercise 2** Check for which values of  $\sigma, \mu, a, b$   $\tilde{S}_t = S_t/X_t$  and  $\tilde{X}_t = X_t/S_t$  are respectively martingales, submartingales, supermartingales w.r.t. the probability measure  $P$ .

**Exercise 3** Use Girsanov Theorem to find a risk-neutral measure  $Q' \sim P$  under which  $\tilde{X}_t$  is a  $Q'$ -martingale. Is this risk neutral measure unique ?

Find also a risk-neutral measure  $Q'' \sim P$  under which  $\tilde{S}_t$  is a  $Q''$ -martingale.

**Exercise 4** Choose a numeraire a corresponding risk-neutral measure for pricing as in exercise 3 and compute an arbitrage free price of the swap option  $F(\omega) = (X_T - S_T)^+$  which gives to the owner the right to change one  $S$  asset with one  $X$  asset at maturity

time  $T$ .

Is the price unique or it depends on the choice of the risk neutral pricing measure ? Compute the set of all arbitrage free prices for  $F$ .

**Exercise 5** Suppose that introduce a third riskless asset  $dR_t = R_t r dt$ ,  $R_0 = 1$ ,  $r \in \mathbb{R}$ . Now we have a market with three instruments,  $S_t, X_t, R_t$ .

Choose  $R$  as a numeraire. Show that there is an unique risk neutral measure  $Q \sim P$  and compute the unique fair price and the hedging strategy of the swap option  $F(\omega)$ .