Mathematical Finance , Solutions -4 (01.10.08), Fall 2008.

Consider the one period market with two assets $(S^0(\omega), S^1(\omega)) > 0$ at time $t = 1$, and initial prices (π^0, π^1) at time $t = 0$, on a probability space $(\Omega, \mathcal{F}, P)$.

We assume that $\pi_0 = \pi_1 = 1$, and we assume that both S^0 and S_1 with positive probabilities take only the two values $\{1/2, 5/3\}$, that is

$$1 > P(S^i = 1/2) = 1 - P(S^i = 5/3) > 0, \text{ for both } \quad i = 0, 1$$

We assume also that S^0 and S^1 are independent and identically distributed under P, which means that

$$P(S^0 = u, S^1 = v) = P(S^0 = u)P(S^1 = v)$$

for all values u, v.

1) Compute the set of risk neutral measures $Q \sim P$.

We remark that since under the measure P the two assets S^0, S^1 are independent, each takin two possible values, in principle there are four possible state of the world, $\Omega = \{\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22}\}$, such that $0 < P(\omega_{ij}) < 1$ for all i, j. Denoting the possible values $s_1 = 1/2, s_2 = 5/3$, we have $S^0(\omega_{ij}) = s_i$ and $S^1(\omega_{ij}) = s_j$.

Choose S^0 as numeraire, look at the discounted asset values, $\tilde{S}^1(\omega) := S^1(\omega)/S^0(\omega)$, we have

$$\begin{aligned} \tilde{S}^1(\omega_{11}) &= 1, \\ \tilde{S}^1(\omega_{12}) &= 10/3, \\ \tilde{S}^1(\omega_{21}) &= 3/10. \\ \tilde{S}^1(\omega_{22}) &= 1. \end{aligned}$$

We compute the set of risk neutral measures $Q \sim P$, denote $q_{ij} = Q(\{\omega_{ij}\})$.

Note that if only the states ω_{12} and ω_{21} where possible, setting $q = q_{12} = 1 - q_{21}$ we would get

$$\begin{aligned} &\tilde{S}^1(\omega_{12})q_{12} + \tilde{S}^1(\omega_{21})q_{21} = \frac{\pi^1}{\pi^0} = \tilde{\pi}^1 = 1 \\ &10/3q + (1 - q)3/10 = 1q_{12} = q = 21/91, q_{21} = (1 - q) = 70/91 \end{aligned}$$

In the other cases $\tilde{S}^1(\omega_{11}) = \tilde{S}^1(\omega_{22}) = \tilde{\pi}^1 = 1$. Therefore of risk neutral measure is given by all possible convex combinations

$$\mathcal{Q} = \{Q = (q_{11}, q_{12}, q_{21}, q_{22}) = (\alpha\beta, (1-\alpha)21/91, (1-\alpha)70/91, \alpha(1-\beta)) : \alpha, \beta \in (0,1)\}$$

2) Show that the market is free of arbitrage but not complete.

The market is arbitrage free but not complete since there is whole two parameter family of risk neutral measures.

3) Consider the european call option $X(\omega) = (S^1(\omega) - 2/3)^+$. Compute the set of arbitrage free prices $\mathcal{C}(X)$ for the contract X.

Consider the discounted option $\tilde{X}(\omega) = X(\omega)/S^0(\omega)$ We have

$$\begin{aligned}
&\tilde{X}(\omega_{11}) = 0, \\
&\tilde{X}(\omega_{12}) = (5/3 - 2/3)/(1/2) = 2, \\
&\tilde{X}(\omega_{21}) = 0 \\
&\tilde{X}(\omega_{22}) = (5/3 - 2/3)/(5/3) = 3/5.
\end{aligned}$$

The set of prices is given by

$$\begin{aligned}
&\mathcal{C}(X) = \{E_Q(\tilde{X}) : Q \in \mathcal{Q}\} \\
&\{((1-\alpha)21/91 \times 2 + \alpha(1-\beta) \times 3/5) : \alpha, \beta \in (0,1)\}
\end{aligned}$$

now since $21/91 \times 2 = 0.4615 < 0.6 = 3/5$, since by the general theory $\mathcal{C}(X)$ is an open interval, we have

$$\mathcal{C}(X) = (0.4615, 0.6)$$

4) Find a superhedging strategy for the claim X, that is a strategy η such that P-almost surely the *consumption* is non-negative $D(\omega) := (\eta \cdot S(\omega) - X(\omega)) \geq 0$.

The superhedging strategy $\eta = (\eta^0, \eta^1)$ must cover the value of the option X at time $t = 1$ in all cases, and by the general theory we know that the value at time $t = 0$ of the cheapest superhedging strategy is $c^+(X) = \sup \mathcal{C}(X) = 0.6$. We get two equations

$$\eta_0\pi^0 + \eta_1\pi^1 = \eta_0 + \eta_1 = 0.6, \eta_0 S^0(\omega_{12}) + \eta_1 S^1(\omega_{12}) = X(\omega_{12})$$

Solving this we obtain

$$\eta^1 = 0.6 - \eta^0 \eta^0 \frac{1}{2} + \eta^1 5/3 = 1,$$

which gives $\eta_0 = 0, \eta_1 = 3/5 = 0.6$.

The consumption process is $D(\omega) = (V(\omega) - X(\omega))$,

where $V(\omega) = \eta^0 S^0(\omega) + \eta^1 S^1(\omega)$. We find that $D(\omega) \geq 0$, since

$$D(\omega_{11}) = D(\omega_{21}) = (3/5)(1/2) - 0 = 0.3 > 0,$$
$$D(\omega_{12}) = D(\omega_{22}) = (3/5)(5/3) - 1 = 0$$

Note that such superhedging strategy gives an arbitrage opportunity for the seller of the option X at price $c^+(X)$.

Consider now the extended market where we added one riskless investment possibility , with value $b = 1$ at time $t = 0$, and deterministic value $B(\omega) = 6/5$ at time $t = 1$.

5) Show that the extended market (B, S^0, S^1) with intial prices (b, π^0, π^1) is complete and free of arbitrage.

There is a mistake here the market will not be complete, since we will have a market with 3 instruments and 4 possible state of the worlds, so the dimenension of the linear space generated by the assets cannot match the dimension of the space of random variables $L^0(\Omega)$.

We must show that the market is still free of arbitrage.

We can argue that the interest rate of the riskless interest B is $1/5$ while the random returns of the assets S^0, S^1 take both values in the set $\{-1/2, 2/3\}$ Since is arbitrage free since $-1/2 < 1/5 < 2/3$ and all the combinations of the values of the random returns $(R^0(\omega), R^1(\omega))$ have positive probability, it is clear that the market is arbitrage free.

We use the geometric argument; choosing the riskless asset B as numeraire, the initial discounted price vector $(1, 1)$ should be in the relative interior of the convex hull of the support of the distribution of the discounted asset vector $(S^1(\omega)/B, S^2(\omega)/B)$.

Now the support of this distribution is formed by 4 points:

$$\{(S^1(\omega)/B, S^2(\omega)/B) : \omega \in \Omega\} = \{(25/18, 25/18), (25/18, 3/25), (3/25, 25/18), (3/25, 3/25)\}$$

and we see the interior of the convex hull of this set is an open square which contains the point $(1,1)$, which was the initial discounted asset price vector. Therefore the market is arbitrage free.

We compute the set of risk neutral measure. We continute using the asset S^0 as numeraire, so that the discounted asset $\tilde{B}^0 = B^0/S^0(\omega)$ will have price 1 at time $t=0$, and will take values

$$\tilde{B}(\omega_{11}) = \tilde{B}(\omega_{12}) = (1/5)/(1/2) = 2/5 = 0.4\tilde{B}(\omega_{21}) = \tilde{B}(\omega_{22}) = (1/5)/(5/3) = 3/25 = 0.12$$

Now we ask that $\alpha, \beta \in (0,1)$ such that

$$1 = E_Q(\tilde{B}) = (\alpha\beta + (1-\alpha)21/91)(2/5) + \alpha(1-\beta) + (1-\alpha)70/91)(3/25)$$

We can fix $\alpha \in (0,1)$ and get one linear equation for β, with solution $\beta = \beta(\alpha) = (265 + 21\alpha)/(91\alpha)$.

Now to get a solution $\beta \in (0,1)$ we solve the equation $\beta(\alpha_0) = 0$ and $\beta(\alpha_1) = 1$ which gives $\alpha_0 = 21/265 = 0.0792 \in (0,1)$ and $\alpha_1 = 53/14 = 3.7857 > 1$.

It follows that the parameter pairs $(\alpha, \beta(\alpha))$ with $\alpha \in (21/265, 1)$, correspond to risk neutral measures $Q(\alpha, \beta(\alpha))$, and we see that the model is arbitrage free as we knew but it is not complete.

6) Compute the price and the hedging strategy of the contract X in the extended market (B, S^0, S^1).

Although the extended market is not complete, the contract $X = (S^1 - 2/3)^+$ depends only on the value of the stock S^1, and we known thta the market (B, S^1) is complete, since we have two instruments and two possible states of the world at time $t = 1$. Therefore all contingent claims which are function of the asset S^1 only are replicable by using the asset S^1 and the bank account B only.

Since we will not use the asset S^0 we have to choose another numeraire.

What we shall do is to choose B as numeraire and find the unique risk neutral measure Q satisfying

$$E_Q(S^1/B) = E_Q(S^1)/B = 1$$

where 1 is the discounted asset price at time $t = 0$.

Setting $S^1(\omega_1) = 1/2$, $S^1(\omega_2) = 5/3$, $q = Q(\omega_1) = 1 - Q(\omega_2)$ we find

$$1 = E_Q(S^1/B) = q(1/2)/(6/5) + (1-q)(5/3)/(6/5)$$

which gives unique solution $q = 2/5 \in (0, 1)$.

The unique price of the option X in this extended market is

$$c(X) = E_Q(X/B) = E_Q(X)/B = 0 \times q + (5/3 - 2/3)^+/(6/5) \times (1 - q) = (5/6) * (3/5) = 1/2$$

The hedging strategy is found by solving the system

$$\eta^0 B + \eta^1 S^1(\omega) = X(\omega) \quad \omega = \omega_1, \omega_2$$

which gives

$$\eta^0 6/5 + \eta^1 1/2 = 0$$
$$\eta^0 6/5 + \eta^1 5/3 = 1$$

with solutions $\eta_0 = -15/42 =$, $\eta_1 = 6/7$.